MS: Do any 4 of the following 7 problems
Ph.D.: Do any 6 of the following 7 problems.

1. Consider the Dirichlet problem in a bounded domain $\mathcal{D} \subset \mathbb{R}^N$ with smooth boundary $S$,

$$\Delta u + a(x)u = f(x), \quad x \in \mathcal{D},$$

$$u|_S = 0, \quad x \in S.$$

(a) Assuming that $|a(x)|$ is small enough, prove the uniqueness of the classical solution.
(b) Prove the existence of the solution in the Sobolev space $H^1(\mathcal{D})$ assuming that $f \in L^2(\mathcal{D})$.
Note: Use Poincare inequality.

2. Consider the Cauchy problem

$$\frac{\partial u}{\partial t} - \Delta u + u^2(x,t) = f(x,t), \quad x \in \mathbb{R}^N, \quad 0 < 0 < T,$$

$$u(x,0) = 0.$$

Prove the uniqueness of the classical bounded solution assuming that $T$ is small enough.

3. Consider the following problem (so called Goursat problem):

find the solution of the equation
\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a(x, t)u = 0 \]
in the square \( \mathcal{D} \), satisfying the boundary conditions
\[ u|_{\gamma_1} = \varphi, \quad u|_{\gamma_2} = \psi, \]
where \( \gamma_1, \gamma_2 \) are two adjacent sides of \( \mathcal{D} \). Here \( a(x, t), \varphi \) and \( \psi \) are continuous functions. Prove the uniqueness of the solution of this Goursat problem.

4. Consider the following functional
\[ F(v) = \int \int \int_{\mathcal{D}} \left[ \sum_{j=1}^{3} \left( \frac{\partial v_j}{\partial x_k} \right)^2 + \alpha \left( \sum_{j=1}^{3} v_j^2(x) - 1 \right)^2 \right] dx, \]
where \( x = (x_1, x_2, x_3) \in \mathbb{R}^3, \ v(x) = (v_1(x), v_2(x), v_3(x)), \ \mathcal{D} \) is a bounded domain in \( \mathbb{R}^3 \) with a smooth boundary \( S \), and \( \alpha > 0 \) is a constant. Let \( u(x) = (u_1(x), u_2(x), u_3(x)) \) be the minimizer of \( F(v) \) among all smooth functions satisfying the Dirichlet condition, \( u_k(x) = \varphi_k(x), \ k = 1, 2, 3 \). Derive the system of differential equations that \( u(x) \) satisfies.

5. Consider the eigenvalue problem on the interval \([0, 1], \)
\[ -y''(t) + p(t)y(t) = \lambda y(t), \]
\[ y(0) = y(1) = 0. \]

(a) Prove that all eigenvalues \( \lambda \) are simple.
(b) Prove that there is at most a finite number of negative eigenvalues.

6. Consider the initial boundary value problem
\[ \frac{\partial u(x, t)}{\partial t} - \frac{\partial^2 u(x, t)}{\partial x^2} + au(x, t) = 0, \quad t > 0, x > 0, \]
\[ u(x, 0) = 0, \quad x > 0 \]
\[ u(0, t) = g(t), \quad t > 0, \]
where \( g(t) \) is continuous function with a compact support, and \( a \) is constant. Find the explicit solution of this problem.

7. Consider the following system of ODEs
\[ u_t = au - buv \]
\[ v_t = -cv + duv \]
in which \( a, b, c, d \) are constants. For the phase plane region \( \mathbb{R}^2^+ = \{(u, v) : u > 0, v > 0\} \), do the following
a) Find all stationary points.
b) Analyze their type.
c) Draw a global picture of the solution set.
e) Show that $\mathbb{R}^2^+$ is an invariant set for this flow.