Ph.D Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2001

MS: Do any 4 of the following 7 problems
Ph.D.: Do any 6 of the following 7 problems.

1. Consider the initial value problem \( u_t = a(u) \) with \( u(0) = u_0 \).
   (i) Work out an example of the function \( a(u) \) for which the solution \( u \) blows up in finite time.
   (ii) Work out an example of the function \( a(u) \) for which the solution \( u \) is not unique.
   (iii) Describe conditions on the function \( a(u) \) so that the solution \( u \) is unique and exists for all time. Justify your answer.

2. Consider the differential operator

\[
L = (d/dx)^2 + 2(d/dx) + \alpha(x)u
\]

in which \( \alpha \) is a real-valued function. The domain is \( x \in [0, 1] \), with Neumann boundary conditions \( du/dx(0) = du/dx(1) = 0 \).
   (i) Find a function \( \phi = \phi(x) \) for which \( L \) is self-adjoint in the norm

\[
||u||^2 = \int_0^1 u^2 \phi dx
\]

   (ii) Show that \( L \) must have a positive eigenvalue if \( \alpha \) is not identically zero and

\[
\int_0^1 \alpha(x)dx \geq 0.
\]

3. Let \( u = u(x, t) \) solve the following PDE in three spatial dimensions

\[
\Delta u = 0
\]

for \( R_1 < r < R(t) \), in which \( r = |x| \) is the radial variable, with boundary conditions \( u(r = R(t), t) = 0 \) and \( u(r = R_1, t) = 1 \). In addition assume that \( R(t) \) satisfies

\[
dR/dt = -\partial u/\partial r(r = R)
\]

with initial condition \( R(0) = R_0 \) in which \( R_0 > R_1 \).
   (i) Find the solution \( u(x, t) \).
   (ii) Find an ODE for the outer radius \( R(t) \).
4. For the ODE
\[ \rho_t = \rho (1 - \rho) \]

do all of the following:
- a) Analyze the type of all stationary points.
- b) Find a conserved energy.
- c) Draw a the phase plane diagram.

5. Consider the system
\[ f_t + f_x = (h^2 - fg) \]
\[ g_t - g_x = (h^2 - fg) \]
\[ h_t = -(h^2 - fg) \]

   a) Find two conserved quantities for this system.
   b) Look for a traveling wave solution in which \((f, g, h) = (f(x - st), g(x - st), h(x - st))\), in which \(|s| < 1\), and find a system of three ODEs for this special solution.
   c) Reduce the system of ODEs for the traveling wave to a single ODE for \(h\).
   d) Show that the resulting ODE has solutions of the form
\[ h = h_0 + h_1 \tanh(\alpha x + x_0) \]
in which \(h_0, h_1, \alpha\) and \(x_0\) are constants.

6. Use the method of characteristics to solve the following partial differential equation in parametric form:
\[ \frac{\partial u}{\partial t} - u \frac{\partial u}{\partial x} = 3u, \quad u(x, 0) = u_0(x). \]

7. Consider the parabolic problem
\[ u_t = u_{xx} + c(x)u \]

for \(-\infty < x < \infty\), in which
\[ c(x) = 0 \quad \text{for} \quad |x| > 1 \]
\[ c(x) = 1 \quad \text{for} \quad |x| < 1. \]

Find solutions of the form \(u(x, t) = e^{\lambda t}v(x)\) in which \(\int_{-\infty}^{\infty} |u|^2 dx < \infty\). (Hint: Look for \(v\) to have the form \(a \exp -k|x|\) for \(|x| > 1\) and \(b \cos \ell x\) for \(|x| < 1\) for some \(a, b, k, \ell\).)