Ph. D. Qualifying Exam
APPLIED DIFFERENTIAL EQUATIONS
Fall 2005

(1) Consider the initial value problem

\[ u_t = v, \quad v_t = |u|^\alpha, \]
\[ u_{t=0} = u_0, \quad v_{t=0} = 0. \]

For what constant values of \( u_0 \geq 0 \) and \( \alpha \geq 0 \) is this problem well-posed, (a) only locally in time or (b) globally in time? Prove your answer.

(2) Consider the two point boundary value operator \( L \) defined for \( u = u(x) \) by

\[ Lu = u'' + u' - a(1 + x^2)u \]

defined on the interval \( x \in [0,1] \) with boundary conditions

\[ u(0) = u(1) = 0 \]

with \( a > 0 \). Let \( \lambda_{a_0} \) be the eigenvalue of smallest absolute value for \( L \) and let \( u_{a_0} \) be the corresponding eigenfunction. Do the following:

(a) Find an inner product in terms of which \( L \) is self-adjoint.
(b) Show that \( \lambda_{a_0} < 0 \).
(c) Show that \( |\lambda_{a_0}| \) is an increasing function of \( a \); i.e., if \( 0 < a_1 < a_2 \), then \( |\lambda_{a_1}| < |\lambda_{a_2}| \).

(3) For the ODE \( f'' - f(f^2 - 1) \) do the following:

(a) Find the stationary points and classify their type.
(b) Find all periodic orbits and all orbits that connect stationary points.
(c) Draw a picture of the phase plane.

(4) Consider the heat equation

\[ u_t = u_{yy} \]
on the real line with initial data \( u_0 = 1, \ y < 0, \ u_0 = 0, \ y > 0 \). (a) Show that the solution \( u(y, t) \) satisfies \( \lim_{t \to \infty} u(y, t) = 1/2 \). (b) Is the limit uniform in \( y \)? Prove your answer.
(5) The Cahn-Hilliard equation for phase separation of a binary alloy is
\[ u_t + \Delta(\epsilon u - \frac{1}{\epsilon} W'(u)) = 0, \]
Where \( W(u) \) is a smooth function of \( u \). Show that
\[ E(u) = \epsilon \frac{1}{2} \int |\nabla u|^2 dx + \frac{1}{\epsilon} \int W(u) dx. \]
is a monotonically decreasing quantity for smooth solutions of the Cahn-Hilliard equation on the torus \( T^2 \).

(6) Let \( f \) be a smooth function defined on \( R^3 \) and suppose that \( \Delta \Delta f = 0 \) for \( |x| \leq a \). Show that
\[ (4\pi a^2)^{-1} \int_{|x|=a} f(x) ds = f(0) + \frac{a^2}{6} \Delta f(0). \]
Hint: Do this first for spherically symmetric \( f \); i.e., for \( f(x) = f(r = |x|) \), for which \( \Delta = r^{-2} \partial_r (r^2 \partial_r) \).

(7) Find the (entropy) solution for all time \( t > 0 \) of the inviscid Burgers equation
\[ u_t + \frac{1}{2} (u^2)_x = 0 \]
with initial condition
\[ u(x, 0) = \begin{cases} 0, & x < -1 \\ x + 1, & -1 < x < 0 \\ 1 - \frac{1}{2} x, & 0 < x < 2 \\ 0, & x > 2. \end{cases} \]

(8) Consider the "eikonal" equation in \( R^2 \):
\[ \phi_x^2 + \phi_y^2 = 1 \]
in the domain \( 0 < x < 2\pi \) and \( 0 \leq y < \infty \), with periodic boundary conditions in \( x \) and boundary data
\[ \phi(x, 0) = \cos(x). \]
Find a solution in an implicit form.