Qualifying Examination on Applied Differential Equations

Wednesday, January 5 2005, 9.00 a.m.–1.00 p.m.

Solve all of the following 7 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

**Problem 1.** Consider the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial u}{\partial x} = 0, \quad 0 < x < 1, \ t > 0,$$

with the boundary conditions

$$\frac{\partial u}{\partial x}(t, 0) = \frac{\partial u}{\partial x}(t, 1) = 0, \quad t > 0,$$

and initial conditions

$$u(0, x) = e^{-x}(\pi \cos \pi x + \sin \pi x), \quad \frac{\partial u}{\partial t}(0, x) = 0, \quad 0 < x < 1.$$

- Show that a separation of variables in (1) leads to an eigenvalue problem in the variable $x$.
- Determine the eigenvalues and the eigenfunctions for the eigenvalue problem in question.
- Determine a solution to (1) which satisfies the boundary and the initial conditions.

**Problem 2.** Let $\varphi \in C^1(\mathbb{R}^2)$. Solve the following Cauchy problem in $\mathbb{R}^3$,

$$\begin{cases}
    x_1 \partial_{x_1} u + 2x_2 \partial_{x_2} u + \partial_{x_3} u = 3u, \\
    u(x_1, x_2, 0) = \varphi(x_1, x_2).
\end{cases}$$

**Problem 3.** Let $u(x)$ be harmonic in the unit disc $|x| < 1$ in $\mathbb{R}^2$, and assume that $u \geq 0$. Prove the following Harnack's inequality:

$$\frac{1 - |x|}{1 + |x|} u(0) \leq u(x) \leq \frac{1 + |x|}{1 - |x|} u(0), \quad |x| < 1.$$
Problem 4. Let \( u(x, t) \in C^\infty(\mathbb{R}^3 \times \mathbb{R}) \) solve the Cauchy problem for the wave equation

\[
\begin{cases}
(\partial_t^2 - \Delta_x) u = 0, & x \in \mathbb{R}^3, \ t > 0, \\
u|_{t=0} = \varphi(x), & \partial_t u|_{t=0} = \psi(x),
\end{cases}
\tag{2}
\]

with \( \varphi(x) \) and \( \psi(x) \) being smooth compactly supported functions on \( \mathbb{R}^3 \). Use an explicit formula for the solution of (2) (the Kirchhoff's formula), to show that there exists a constant \( C' > 0 \) such that we have, uniformly in \( x \in \mathbb{R}^3 \),

\[
|u(x, t)| \leq \frac{C'}{t}, \quad t > 0.
\]

Problem 5. Solve the inhomogeneous problem for the Laplace operator in the unit disc \( D = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 < 1\}, \)

\[
\begin{cases}
\Delta u = x^2 - y^2 \quad \text{in } D \\
u = 0 \text{ along } \partial D.
\end{cases}
\]

Problem 6. Find the Fourier transform of the integrable function \( x \mapsto (\sin x)^2/x^2 \).

*Hint.* Determine first the Fourier transform of \( x \mapsto x^{-1} \sin x \).

Problem 7. Consider an autonomous system in \( \mathbb{R}^n \), \( x'(t) = f(x(t)) \), where \( f = (f_1, f_2, \ldots, f_n) \) is a smooth vector field, such that

\[
\sum_{k=1}^{n} x_k f_k(x) < 0 \quad \text{for } x \neq 0.
\]

Show that \( x(t) \to 0 \) as \( t \to \infty \), for each solution of the system, independently of the initial condition \( x(0) \).