Please solve all 8 problems.

1. Let $\phi(x)$ be continuous and bounded in $\mathbb{R}^n$. Assume that $\lim_{|x| \to \infty} \phi(x) = \phi_0$. Consider the Cauchy problem

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = 0 \quad \text{for } 0 \leq t, x \in \mathbb{R}^n$$

$$u(x,0) = \phi(x).$$

Prove that $\lim_{t \to \infty} u(x,t) = \phi_0$.

2. Let $A_i(x)$, $i = 1, 2$, be smooth functions in a bounded domain $\Omega \subset \mathbb{R}^n$ such that $A_1 = A_2$ on $\partial \Omega$. Assume that

$$\Delta A_1 + \sum_{j=1}^n \left( \frac{\partial A_1}{\partial x_j} \right)^2 = \Delta A_2 + \sum_{j=1}^n \left( \frac{\partial A_2}{\partial x_j} \right)^2$$

in $\Omega$. Prove that $A_1(x) = A_2(x)$ in $\Omega$.

3. Let $S$ be a strip $\{0 < x_1 < a, -\infty < x_2 < \infty \}$. Let $u(x_1, x_2)$ be a smooth solution of $\Delta u + \lambda u = 0$ in $S$ satisfying the boundary conditions $u(0, x_2) = 0$, $u(a, x_2) = 0$, $-\infty < x_2 < \infty$. Here $\lambda$ is a real constant. Prove that if $\int_S |u(x_1, x_2)|^2dx_1dx_2 < \infty$, then $u(x_1, x_2) = 0$ in $S$.

4. Consider the initial boundary value problem:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + 2 \frac{\partial^2 u(x,t)}{\partial x \partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + a(x,t) \frac{\partial u(x,t)}{\partial x} = 0 \quad (1)$$

for $0 \leq t < \infty$, $-\infty < x < \infty$ with

$$u(x,0) = f(x), \quad \frac{\partial u(x,0)}{\partial t} = g(x) \quad (2)$$

for $-\infty < x < \infty$, where $f(x), g(x)$ are smooth functions having compact supports and $a$ is a smooth bounded function. Find an estimate for the solution of (1), (2) that will imply uniqueness.
5. Consider the initial value problem

\[ \frac{du}{dt} = cu^{1+\alpha} \]
\[ u(0) = u_0 \]

in which $c > 0$ and $\alpha > 0$ are constants and $0 < u_0 < 1$.

(a) Find the solution of this ODE.
(b) Find the blowup time $t_*$ at which $u \to \infty$.
(c) Find the value of $\alpha$ that minimizes $t_*$ for fixed values of $c$ and $u_0$.

6. Let $u = u(x, t)$ in which $u \in R^2$ and $x \in R^2$. Solve the following problem by the method of characteristics

\[ \frac{\partial u}{\partial t} + u \cdot \nabla u = u \]
\[ u(x, 0) = x. \]

Note that the $j$th component of $u \cdot \nabla u$ is

\[ (u \cdot \nabla u)_j = \Sigma_{i=1}^2 u_i \partial_{x_i} u_j. \]

7. Let $u$ and $\lambda$ be the eigenfunction and eigenvalue of the two point boundary value problems on $0 \leq x \leq L$

\[ u_{xx}(x) - a(x)u(x) = -\lambda u(x) \]
\[ u(0) = u(L) = 0 \]

in which $\lambda$ and $L$ are constants. Assume that $\lambda$ is the lowest eigenvalue for this problem.

(a) Show that $a > 0$ implies $\lambda > 0$.
(b) Find an example showing that $a < 0$ does not imply $\lambda < 0$.
(c) Show that $\lambda$ is a decreasing function of $L$.

8. For $i = 1, 2$ and $0 \leq t \leq T$, let $\Omega_i(t)$ be an open smooth bounded domain in $R^2$ for each $t$ with $\Omega_1(0) = \Omega_2(0)$ and $\partial \Omega_1(t) \subset \Omega_2(t)$ for
$0 < t \leq T$ (i.e., $\Omega_1(t)$ is strictly contained in $\Omega_2(t)$ for $t = 0$). Let $u_i$ for $i = 1, 2$ solve

\[
\frac{\partial u_i}{\partial t} - \Delta u_i = 0 \quad \text{for } x \in \Omega_i(t) \text{ and } 0 \leq t \leq T
\]

\[
u_i(x, 0) = f(x) \quad \text{for } x \in \Omega_i(0)
\]

\[
u_i(x, t) = 0 \quad \text{for } x \in \partial \Omega_i(t)
\]

in which the initial data $f$ is independent of $i$ with $f > 0$ in $\Omega_i(0)$.

(a) Show that $u_i > 0$ for $x \in \Omega_i(t)$ and $0 < t \leq T$

(b) Show that $u_1 < u_2$ for $x \in \Omega_1(t)$ and $0 < t \leq T$. 