Applied Differential Equations – Fall 2008

Each problem is worth 10 points.

1. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary \( \Gamma \). Let \( f \) be a continuous function in \( \Omega \) and \( g \) be a continuous function on \( \Gamma \).

(a) Consider the functional

\[
J[u] = \int_{\Omega} (|Du|^2 + fu)dx + \int_{\Gamma} gu^2d\sigma
\]

applied to smooth functions \( u \) defined in \( \Omega \cup \Gamma \). Determine the differential equation and boundary condition satisfied by a function which minimizes \( J \).

(b) State and prove conditions for uniqueness of solutions to the boundary value problem

\[-\Delta u = f \text{ in } \Omega, \quad u_\nu + gu = 1 \text{ in } \Gamma,\]

where \( \nu \) denotes the normal vector at each point on \( \Gamma \), outward with respect to \( \Omega \).

2. Let \( g : [0, \infty) \to \mathbb{R} \) be a continuous function with \( g(0) = 0 \). Derive an integral formula for the solution of the problem

\( u_t - u_{xx} = 0 \) in \( \mathbb{R}^+ \times (0, \infty) \), \( u = 0 \) on \( \mathbb{R}^+ \times \{ t = 0 \} \), and \( u = g \) on \( \{ x = 0 \} \times [0, \infty) \)

in terms of \( g \). Consider the function \( v(x,t) = u(x,t) - g(t) \) extended to \( \mathbb{R} \times \mathbb{R}^+ \) by \( v(x,t) = -v(-x,t) \).

3. Consider the initial value problem for Burger’s equation

\( u_t + u_x u = 0 \) in \( \mathbb{R} \times (0, \infty) \), \( u = g \) on \( \mathbb{R} \times \{ t = 0 \} \).

Find the entropy solution of this problem with the initial data

\[
g(x) = \begin{cases} 
0 & \text{if } x > 1, \\
1 - x & \text{if } 0 < x < 1, \\
1 & \text{if } x < 0,
\end{cases}
\]

Also find the maximal time interval \([0, t^*)\) on which the solution is continuous.

4. A traveling wave solution of speed \( c \) to \( u_t = u_{xx} + 1 - u^2 \) is a solution of the form \( u(x,t) = f(x - ct) \). Using phase plane analysis, explain how this equation has a unique traveling wave solution of speed \( c \) with \( \lim_{x \to -\infty} f(x) = 1 \) and \( \lim_{x \to -\infty} f(x) = -1 \) as long as \( c > 0 \). A rigorous argument is not asked for here. Then prove that the function \( f(x) \) will not be monotonic decreasing when \( c < 2\sqrt{2} \).
5. Suppose that $f(x)$ is a continuous function such that $f(x) \equiv 0$ when $|x| > R$. Show that
\[
u(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|} \, dy\]
is a 'weak solution' to $\Delta u = f$ in the sense that
\[
\int u \Delta \phi \, dx = \int f \phi \, dx
\]
for all $\phi \in C^2(\mathbb{R}^3)$ satisfying $\phi(x) \equiv 0$ for $|x| > R + 1$.

6. Consider the first order system of equations
\[
u_t + \sum_{j=1}^{n} A_j \nu x_j = 0,
\]
where $u(x, t) = (u_1(x, t), \ldots, u_m(x, t))$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and the $A_j$'s are symmetric $m \times m$ matrices with constant real entries. Use an energy argument to show that the domain of dependence of $(x_0, t_0)$, $t_0 > 0$, for a solution of the system (1) is contained in the cone
\[
\{|x - x_0| \leq \Lambda(t_0 - t)\}
\]
where $\Lambda = \max_{\{|\xi| = 1, 1 \leq l \leq m\}} |\lambda_l(\xi)|$, and $\lambda_l(\xi)$, $l = 1, \ldots, m$, are the eigenvalues of the matrix $A(\xi) = \sum_{j=1}^{n} \xi_j A_j$.

7. Suppose $u$ is a smooth solution of the following problem
\[
u_{xxx} + u_{xx} - u^3 = 0 \text{ in } [0, 1] \times (0, \infty), \quad u(0, t) = u(1, t) = 0 \text{ for } (0, \infty)
\]
with initial data $u(x, 0) = x(x - 1)$. Derive a differential inequality for $w(t) := \int_0^1 (u_x)^2(x, t) \, dx$, and show that $u(x, t)$ uniformly tends to zero as $t \to \infty$.

8. Suppose that $q(x)$ is a real-valued continuous function such that $\int_0^1 q(x) \, dx = 0$, but $q(x)$ is not identically zero. Show that $Lu = -u'' + q(x)u$ with the boundary conditions $u'(0) = u'(1) = 0$ must have a strictly negative eigenvalue by showing that $\int_0^1 uLudx$ can be negative.