Qualifying Exam on Applied Differential Equations

Tuesday, September 15, 2009, 2:00 p.m.–6:00 p.m.

Solve the following 8 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

Problem 1. Let \( u(x) \) be harmonic in the open ball \( \{ x \in \mathbb{R}^n; |x| < R \} \). Assume that \( u(x) \) is bounded. Show that the following Harnack inequality holds,

\[
\frac{R^2 - |x|^2}{(R + |x|)^n} u(0) \leq R^{2-n} u(x) \leq \frac{R^2 - |x|^2}{(R - |x|)^n} u(0), \quad |x| < R.
\]

Problem 2. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( V \in C(\overline{\Omega}) \). Show that for \( \varepsilon > 0 \) small enough, the Dirichlet problem

\[
(-\Delta + \varepsilon V) u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{along} \quad \partial \Omega
\]

has a unique solution in the space \( H^1_0(\Omega) \), for each \( f \in L^2(\Omega) \).

Problem 3. Consider the equation (sometimes called the Beltrami equation)

\[
\frac{\partial u}{\partial x_1} + i \frac{\partial u}{\partial x_2} - \mu \left( \frac{\partial u}{\partial x_1} - i \frac{\partial u}{\partial x_2} \right) = \frac{\partial g}{\partial x_1} + i \frac{\partial g}{\partial x_2}.
\]

For each smooth function \( g \) of compact support this equation is supposed to determine a smooth square-integrable function \( u \). Show that it does that provided that the complex number \( \mu \) satisfies \( |\mu| \neq 1 \), and one has the estimate

\[
||u||_{L^2(\mathbb{R}^n)} \leq \frac{1}{||\mu|-1||} ||g||_{L^2(\mathbb{R}^2)}.
\]

Problem 4. Let \( Lu = -u_{xx} + V(x)u \), where \( V(x) \) is real-valued, and \( A u = 4u_{xxx} - 3((V u)_x + V u_x) \). A page of exciting computations shows that the commutator \( LA - AL \) is given by

\[
(LA - AL)u = (6VV_x - V_{xxx})u.
\]

Do not do that computation during this examination. Instead suppose that \( V \) depends on the parameter \( t \) as well as \( x \), and is a solution of the evolution equation \( V_t = 6VV_x - V_{xxx} \) (the Korteweg-De Vries equation). Suppose that \( u(x, t) \) satisfies

\[
L(t)u = -u_{xx} + V(t)u = \lambda(t)u \quad \text{and} \quad \int_{\mathbb{R}} u^2(x, t) dx = 1,
\]

i.e. \( u(x, t) \) is a normalized eigenfunction for the operator \( L(t) \). Show that \( \lambda(t) \) must be independent of \( t \).
Problem 5. Solve the Hamilton-Jacobi equation,

\[
\begin{aligned}
&\left\{ u_t + \frac{1}{2} (u_x)^2 - x = 0, \\
&u(x, 0) = \alpha x, \quad \alpha \in \mathbb{R}.
\end{aligned}
\]

The solution is linear in \(x\), but we want you to use the method of characteristics to solve this problem. The linearity in \(x\) is a check on your answer.

Problem 6. Let \(x(t)\) be a nonnegative differentiable function such that

\[
x'(t) \geq \frac{1}{1 + tx(t)} + t - 1, \quad t \geq 0.
\]

Show that \(x(t) \geq 1 - \exp(-t^2/2)\) for \(t \geq 0\).

Hint. Derive a differential equation for the function \(t \mapsto 1 - \exp(-t^2/2)\).

Problem 7. Let \(u(x, t)\) solve the wave equation

\[
\begin{aligned}
&\left\{ (\partial_t^2 - \Delta) u(x, t) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}, \\
u(x, 0) = \varphi \in C_0^\infty(\mathbb{R}^n), \quad \partial_t u(x, 0) = 0.
\end{aligned}
\]

Show that the function

\[
\tilde{u}(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-s^2/4t} u(x, s) \, ds, \quad t > 0
\]

satisfies the initial value problem for the heat equation,

\[
(\partial_t - \Delta) \tilde{u}(x, t) = 0, \quad t > 0, \quad \tilde{u}(x, 0) = \varphi(x).
\]

This is sometimes called the transmutation formula.

Problem 8. Consider a linear damped wave equation with a constant damping factor \(a \in (0, 1)\),

\[
\begin{aligned}
&\left\{ (\partial_t^2 - \Delta_x + a\partial_t) u(x, t) = 0, \\
u(x, 0) = 0, \quad \partial_t u(x, 0) = f(x).
\end{aligned}
\]

Here \(t \geq 0\) and \(x \in T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2\). Assume also that \(f \in C^\infty(T^2)\).

- Find an explicit formula for the solution of this problem \(u(x, t)\) in terms of Fourier series.

- Show that the energy of the solution,

\[
E(t) = \frac{1}{2} \int_{T^2} (|\nabla_x u|^2 + |\partial_t u|^2) \, dx, \quad t \geq 0
\]

decreases at an exponential rate as \(t \to \infty\). What is the rate of the exponential decay?