Qualifying Exam on Applied Differential Equations

Wednesday, March 25, 2009, 2:00 p.m.–6:00 p.m.

Solve any 7 of the following 9 problems. In doing so, provide clear and concise arguments. Draw a figure when necessary.

**Problem 1.** Let $h(t)$ and $a(t)$ be continuous and bounded functions on $[0, \infty)$, with $a(t) \geq 0$. Let $x(t)$ be a continuous function such that

$$x(t) \leq h(t) \int_0^t a(s)x(s) \, ds + \frac{1}{1 + t^2}, \quad t \geq 0.$$ 

Assume that

$$\int_0^\infty |h(t)| \, dt < \infty.$$ 

Show that $x(t)$ is bounded above on $[0, \infty)$.

**Problem 2.** Let $p(x) \in C^1([0,1])$ and $q \in C([0,1])$ be real-valued with $p(x) > 0$. Show that the eigenvalue problem

$$-\frac{d}{dx} \left( p(x) \frac{du}{dx} \right) + q(x)u = \lambda u, \quad u(0) = u(1) = 0$$

has the following properties:

- All eigenvalues are simple.
- There are at most finitely many negative eigenvalues.

**Problem 3.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let $u \in C^\infty(\Omega)$ be harmonic in $\Omega$, so that

$$\Delta u = 0.$$ 

Show that there exists a constant $C = C(n)$ depending on the dimension $n$ only such that

$$|\nabla u(x)| \leq \frac{C}{d(x)^{n-2}} \sup_{\Omega} |u(x)|, \quad x \in \Omega. \tag{1}$$

Here

$$d(x) = \inf_{y \in \partial \Omega} |x - y|$$

is the Euclidean distance from $x$ to the boundary of $\Omega$. Generalize (1) to obtain similar bounds on the higher order derivatives of $u$.

Hint. Use the Poisson formula for the function $u$ in a ball.
Problem 4. Let \( \Omega \subset \mathbb{R}^n \) be a bounded open set and let \( V \in C(\overline{\Omega}) \) satisfy \( V(x) \geq 0 \). Show that for each \( f \in L^2(\Omega) \), the Dirichlet problem

\[
(-\Delta + V) u = f \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{along} \quad \partial \Omega
\]

has a unique solution in the space \( H^1_0(\Omega) \).

Problem 5. Consider the complementary error function

\[
F(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.
\]

Show that we have, as \( x \to \infty \),

\[
F(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \left( 1 + O \left( \frac{1}{x^2} \right) \right).
\]

Show also that this estimate for large \( x \) can be refined to a complete asymptotic expansion,

\[
F(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{a_k}{x^{2k}},
\]

for some coefficients \( a_k \). (You do not have to determine the \( a_k \)'s).

Problem 6. Consider an initial value problem for the focusing cubic non-linear Schrödinger equation,

\[
iv_t = -\frac{1}{2} v_{xx} - |v|^2 v, \quad u(x, 0) = \phi(x).
\]

Show that the following are conserved quantities for (2) (you may assume that the function \( u(x, t) \) vanishes as \( |x| \to \infty \), together with all of its derivatives).

- Mass

\[
\int_{-\infty}^{\infty} |u(x, t)|^2 \, dx
\]

- Energy

\[
\int_{-\infty}^{\infty} \left( \frac{1}{2} |\partial_x u(x, t)|^2 - \frac{1}{2} |u(x, t)|^4 \right) \, dx.
\]

Hint. The function \( u(x, t) \) in (2) is complex-valued. In the computations, use that \( |u|^2 = u\overline{u} \).

Problem 7. Solve the following PDE,

\[
\begin{cases}
  u_t + u_x^2 = 0, \\
  u(x, 0) = -x^2.
\end{cases}
\]
Find the time $T$ for which $|u| \to \infty$ as $t \to T$.

**Problem 8.** Consider the hyperbolic equation

$$u_{tt} + 3u_{xt} + 2u_{xx} = 0$$  \hspace{1cm} (3)

in the quarter-plane $Q = \{x > 0, t > 0\}$. Assign boundary conditions along $t = 0$ and $x = 0$ such that the boundary value problem for (3) in $Q$ will have a unique solution.

**Problem 9.** Consider the boundary value problem in a smooth bounded domain $D$ in $\mathbb{R}^n$,

$$\Delta u = 0 \quad \text{in } D, \quad \frac{\partial u}{\partial n} + a(x)u = f \quad \text{on the boundary of } D.
$$

Here $n$ is outer unit normal to $\partial D$.

- Find a functional whose Euler-Lagrange equation leads to the boundary value problem above.

- Assume that $a(x) > 0$. Prove that this boundary value problem has a unique smooth solution.