1. Determine the constants $A$ such that the differential equation

$$\frac{d^2 u}{dx^2} + u = A + x$$

has a solution satisfying $u(0) = u(\pi) = 0$.

2. (a) Solve

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x$$

with the initial data $(u(x,0), v(x,0)) = (f(x), g(x))$.

(b) Find all boundary conditions of the form $au(0,t) + bv(0,t) = 0$ which make the initial value problem in part (a) well-posed in $x \geq 0$, $t \geq 0$.

3. Consider the competition with limited resources model

$$\dot{x} = (a_1 - b_1 x - c_1 y)x \quad \dot{y} = (a_2 - b_2 x - c_2 y)y$$

Here $a_i$, $b_i$ and $c_i$ are positive constants with $c_1 a_2 > a_1 c_2$ and $b_2 a_1 > b_1 a_2$. Note that this implies $c_1 b_2 > c_2 b_1$.

a) Find the equilibria of this system in the closed quarter plane $x \geq 0$, $y \geq 0$.

b) Show that an equilibrium in the open quarter plane $x > 0$, $y > 0$ must be a saddle.

c) Make a plausible phase plane diagram for trajectories in the closed quarter plane.

4. Use the method of characteristics to find a solution to

$$u_t + uu_x = -x, \quad t \geq 0$$

with $u(x,0) = f(x)$, $-\infty < x < \infty$. You will not be able to find $u(x,t)$ explicitly. However, if $f'(x) \geq 0$, show that the solution will exist for $t \in [0, \pi/2)$. 
5. Assume that \( y = \phi(x) \) is a smooth, one-to-one mapping of the domain \( D \subset \mathbb{R}^2 \) onto the domain \( \hat{D} \subset \mathbb{R}^2 \). Let \( \phi'(x) \) be the Jacobian matrix of \( \phi \), and assume that \( h(x) = |\det \phi'(x)| \neq 0 \). Use the weak form of the equation to show that the boundary value problem

\[
- \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \left[ \beta(x) \frac{\partial u}{\partial x_i} \right] = f \text{ in } D, \quad u = 0 \text{ on } \partial D
\]

is equivalent to

\[
- \frac{1}{\hat{h}(y)} \sum_{i=1}^{2} \frac{\partial}{\partial y_i} \left[ \sum_{j=1}^{2} \hat{h}(y) \beta_{ij}(y) \frac{\partial \hat{u}}{\partial y_j} \right] = \hat{f} \text{ in } \hat{D}, \quad \hat{u} = 0 \text{ on } \partial \hat{D},
\]

where \( \hat{u}(\phi(x)) = u(x) \), \( \hat{f}(\phi(x)) = f(x) \), \( \hat{h}(\phi(x)) = h(x) \), and you need to find the matrix \( (\beta_{ij}(y)) \).

6. In this problem we have the domains in the \((x_1, x_2)\)-plane

\[
\Omega^a_+ = \{|x - (1,0)| \leq a\} \cap \{x_1 \geq 0\} \quad \text{and} \quad \Omega^a_- = \{|x - (-1,0)| \leq a\} \cap \{x_1 \leq 0\},
\]

and set \( \Omega^a = \Omega^a_+ \cup \Omega^a_- \). Consider the Neumann problem

\[
\Delta u = f, \quad x \in \Omega^a, \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega^a,
\]

where \( \int_{\Omega^a_+} f dx = 1 \) and \( \int_{\Omega^a_-} f dx = -1 \).

(a) Prove the existence of a solution to this Neumann problem when \( a > 1 \) and the nonexistence of a solution when \( 0 < a < 1 \).

(b) Show that \( \max_{\Omega^a} |\nabla u| \to \infty \) as \( a \downarrow 1 \). Note that the length of the line segment \( L = \Omega^a_- \cap \Omega^a_+ \) goes to zero as \( a \downarrow 1 \).

7. Consider the heat equation, \( u_t - \Delta u = 0 \), in a bounded domain \( D \) in \( \mathbb{R}^n \) with the initial condition \( u(x, 0) = 0 \) and the boundary condition \( u(x, t) = f(x) \) on \( \partial D \). Find an expansion for the solution to this problem in terms of eigenfunctions of \( \Delta \) and the solution of the Dirichlet problem \( \Delta w = 0 \) in \( D \), \( w = f \) on \( \partial D \). What is leading term in the asymptotic expansion of \( u(x, t) - w(x) \) as \( t \to \infty \)?

8. Let \( u(x, t) \) be the solution to

\[
u_{tt} + a^2(x, t) u_t - \Delta u = 0 \text{ in } D, \quad u(x, t) = 0 \text{ on } \partial D
\]

with \( (u(x, 0), u_t(x, 0)) = (f(x), g(x)) \). Prove that \( \int_D u^2(x, t) dx \) is bounded for \( t \in [0, \infty) \). You may assume that \( D \) is a bounded domain with smooth boundary, \( f \) and \( g \) are smooth functions vanishing on \( \partial D \), and that \( a \) is a smooth function on \( D \times [0, \infty) \).