ADE Exam, Fall 2011
Please do All 8 problems.

1. Let $a, b$ be two points in $\mathbb{R}^2$ and consider $U(x): (\mathbb{R}^2 - \{a, b\}) \to \mathbb{R}$ be a smooth function which satisfies $\lim_{q \to \infty} \sup_{|U(q)|} = 1$. Let us consider a system of ODEs for $(p(t), q(t)) \in \mathbb{R}^2 \times (\mathbb{R}^2 - \{a, b\})$:  
\[
\begin{align*}
    p(t) &= \nabla U(q(t)) \\
    q(t) &= p(t),
\end{align*}
\]
with initial data $p(0) \in \mathbb{R}^2$ and $q(0) \in \mathbb{R}^2 - \{a, b\}$.

(a) Show that if $0 < T < \infty$ and if $(p(t), q(t))$ is a solution defined on $[0, T)$ then 
\[\sup_{t \in [0, T)} |p(t)|, |q(t)| < \infty.\]

(b) Let $[0, T)$ be the maximal interval of existence for $(p(t), q(t))$ with $T < \infty$. Show that $\lim_{t \to T} q(t)$ exists, and moreover 
\[\lim_{t \to T} q(t) = a \text{ or } b.\]

2. Let $\mathbf{v}(x): \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ vector field. Let $\theta(x, t): \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a smooth function solving the following equation: 
\[\theta_t = \Delta(\theta^2) + \nabla \cdot (\mathbf{v}\theta),\]
and $\theta(x, 0)$ is bounded from above and below.

(a) Show that $\theta$ stays bounded, both from above and below, for all times $t \geq 0$ if $\nabla \cdot \mathbf{v} = 0$ for all times.

(b) Now suppose that $|\nabla \cdot u(x)| \leq M$ for all $x \in \mathbb{R}^n$. If $\theta(x, 0) \leq 1$, show that $\theta(x, t) \leq e^{Mt}$ for all $t > 0$.

3. Suppose $u(x, t): \mathbb{R}^n \to [0, T) \to \mathbb{R}$ is a smooth solution of 
\[u_t(x, t) = \Delta u + u, \quad u(x, 0) = u_0(x) \in L^2(\mathbb{R}^n),\]
with 
\[\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} = M < \infty.\]
(a) Present $u(x,t)$ in the form of

$$u(x,t) = u_0(x) + \int_0^t \int_{\mathbb{R}^n} G(x-y)u(y,s)dyds$$

for $0 \leq t \leq T$.

(b) Show that the integral formula in (a) is well-defined in $L^2(dx)$ by showing that

$$\| \int_{\mathbb{R}^n} G(x-y)u(y,s)dyds \|_{L^2(dx)} \leq M,$$

for any $0 \leq s \leq T$.

(c) Using (b), evaluate $M$ in terms of the $L^2$-norm of $u_0(x)$.

4. Show that all solutions of the nonlinear heat equation

$$u_t = \Delta(u^4) \text{ in } |x| < 1, \quad u = 0 \text{ on } |x| = 1$$

vanishes to zero as $t \to \infty$.

5. Consider the initial value problem

$$u_t + (f(u))_x = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x,0) = \phi(x).$$

Assume that $f$ is smooth and uniformly convex ($f''(x) > \theta > 0 \forall x$ for some $\theta > 0$)

(a) Show that if $\phi(x) = -x$, then there is a point at which $|u_x| \to \infty$ in finite time.

(b) Consider the Riemann initial data:

$$\phi(x) = \begin{cases} 
  u^-, & x < 0 \\
  u^+, & x > 0
\end{cases}$$

Compute the entropy solution (show that entropy condition is satisfied). Consider both cases: $u^- > u^+$ and $u^- < u^+$.

6. Consider the initial value problem

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$
(a) Use energy methods to prove the value of the solution $u$ at the point $(x_0, t_0)$ depends at most on the values of the initial data in the interval $(x_0 - 3t_0, x_0 + t_0)$.

(b) Use energy methods to prove uniqueness of solutions if the initial data has compact support.

7. Let $\Omega$ be an open, bounded set in $\mathbb{R}^2$. Let $u = \begin{pmatrix} u \\ v \end{pmatrix} \in \Omega$ and let $f : \Omega \to \mathbb{R}^2$, $f = \begin{pmatrix} f_1(u, v) \\ f_2(u, v) \end{pmatrix}$ and consider the problem

$$\frac{du}{dt} = f(u).$$

(a) Show that all stationary points are saddles if $\nabla \cdot f = 0$ and $\nabla \times f = \frac{\partial f_2}{\partial v} - \frac{\partial f_1}{\partial u} = 0$.

(b) Show that $f : \Omega \to \mathbb{R}^2$ must be $C^\infty$ on $\Omega$.

8. Let $\Omega$ be an open set in $\mathbb{R}^n$, and suppose there exists a nonzero function $u$ satisfying

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases}$$

for some $\lambda$.

(a) Show that $\lambda$ is positive.

(b) Let $\lambda_1$ be the smallest positive such $\lambda$. Show that

$$\lambda_1 = \min \left\{ \int_U |Du|^2 dx : u \in H_0^1(U), \int_U u^2(x) dx = 1 \right\}$$