Please solve all 8 problems.

1. Consider the Cauchy problem in $\mathbb{R}^n \times [0, +\infty)$:
   (1) $\frac{\partial u}{\partial t} = t^2 \Delta u(x, t), \ t > 0, \ x \in \mathbb{R}^n$,
   (2) $u(x, 0) = g(x), \ \text{where } g(x) \in L_2(\mathbb{R}^n)$.

   Prove that there exists a weak solution $u(x, t)$ of (1), (2) such that $u(x, t)$ is continuous in $t \geq 0$ with values in $L_2(\mathbb{R}^n)$.

   Prove also that
   $$\|u(x, t) - g(x)\|_{L_2(\mathbb{R}^n)} \to 0$$
   when $t \to +0$. (Hint: Use Plancherel’s Theorem).

2. Let
   $$\frac{\partial^2 u(x, t)}{\partial t^2} - \frac{\partial^2 u(x, t)}{\partial x^2} + C(x, t) u = 0$$
   for $t > 0, \ u(x, 0) = \varphi(x), \ \frac{\partial u(x, 0)}{\partial t} = \psi(x), \ x \in \mathbb{R}^1$.

   Assume $C(x, t), \varphi(x), \psi(x)$ are smooth functions equal to zero for $|x| > R$. Prove that $u(x, t) = 0$ for $|x| > R + t$.

3. Let $D$ be a domain in $\mathbb{R}^n$ with smooth boundaries.
   a) Let $u(x)$ be a $H_1$ solution of
      $$-\Delta u + \frac{x}{3} u = 0 \ \text{in } D,$$
      $$u_{\mid \partial D} = 0.$$

      Prove that $u \equiv 0$ in $D$.
   b) Let $u(x)$ be a $H_1(D)$ solution of
      $$-\Delta u - \alpha u^\frac{1}{3} = 0,$$
      $$u_{\mid \partial D} = 0.$$

      Prove that $u \equiv 0$ if $\alpha > 0$ is small. (Hint: Use Poincare’s Inequality: $\int_D u^2 dx \leq C \int_D |Du|^2 dx$ for all $u \in H_0^1(D)$.)
4.  a) Let
\[ \Delta u - q(x)u = 0 \quad \text{in} \quad \mathbb{R}^n, \]
where \( q(x) \geq 0 \) is bounded.
Suppose \( u(x) \to 0 \) uniformly when \( |x| \to \infty \).
Prove that \( u(x) \equiv 0 \).

b) Find a nontrivial solution of
\[ \Delta u + u = 0 \quad \text{in} \quad \mathbb{R}^3 \]
such that \( u(x) \to 0 \) when \( |x| \to \infty \). (Hint: Consider radial solutions)

5. Consider the following autonomous ODE:
\[ y_1' = y_2, \quad y_2' = -y_1 + (1 - y_1^2 - y_2^2)y_2. \]
Show that any solution \( x(t) = (x_1(t), x_2(t)) \) of above system converges to \((\sin(t + c), \cos(t + c))\) as \( t \to \infty \), for some constant \( c \).

6. Draw the phase space for the competing species system
\[ x' = x(2 - x - y), \quad y' = y(3 - 2x - y). \]
How likely is it that both species survive?

7. Let \( \Omega \) be a connected, bounded domain in \( \mathbb{R}^n \) with smooth boundary, and let \( f(x), g(x) : \mathbb{R}^n \to \mathbb{R} \) be smooth. Show that there is at most one smooth solution of the following equation
\[
\begin{cases}
  u_t - \Delta u + |\nabla u|^2 = 0 & \text{in} \quad \Omega \times (0, \infty) \\
  u = g(x) & \text{on} \quad \partial \Omega \times (0, \infty) \\
  u(x, 0) = f(x) & \text{in} \quad \Omega.
\end{cases}
\]

8. Show that
\[ u(x, t) = -\frac{2}{3}(t + \sqrt{3x + t^2}) \text{ for } 4x + t^2 > 0, \quad u(x, t) = 0 \text{ for otherwise.} \]
is an entropy solution of the equation \( u_t + uu_x = 0 \).