1. Let us consider the continuity equation \( \rho_t + \nabla \cdot (\rho \vec{v}) = 0 \) in \( \mathbb{R}^3 \times (0, \infty) \to \mathbb{R} \) with \( \vec{v}(x) : \mathbb{R}^n \to \mathbb{R}^n \) and initial data \( \rho_0 \).

(a) Represent \( \rho \) in terms of \( \rho_0 \) using the method of characteristics, assuming that \( \vec{v} \) is Lipschitz continuous. Explain where the Lipschitz continuity assumption is used in the argument.

(b) Suppose \( -1 < \nabla \cdot \vec{v} \) in \( \mathbb{R}^3 \) and \( \rho_0 = \chi_{|x|<1} \), where \( \chi_A \) denotes the characteristic function of a set \( A \). Show that then \( \Omega_1 := \{ x : \rho(x, 1) > 0 \} \) has its volume greater than \( 4/3 \).

(Hint: you may use the fact, which follows from your answer for (a), that the solution of \( u_t + \nabla u \cdot \vec{v} = 0 \) shares the same characteristic path as \( \rho \).)

2. Consider the following parabolic equation
\[
\theta_t = \Delta ((|x|^2 + 1)\theta) + |D\theta| - 4n\theta \quad \text{for} \quad (x,t) \in \mathbb{R}^n \times (0, \infty).
\]

(a) Let \( \theta_1(x,t) \) and \( \theta_2(x,t) \) be two smooth, nonnegative solutions of the above equation which vanishes at infinity, with ordered initial data \( \theta_1(x,0) \leq \theta_2(x,0) \). Show that then \( \theta_1(x,t) \leq \theta_2(x,t) \) for all \( t > 0 \).

(b) Let \( \theta \) be a smooth, nonnegative, integrable solution of above equation, where all its derivatives and its products with \( |x|^2 \) vanish at \( |x| \to \infty \). Show that \( \int \theta(\cdot,t)dx \) exponentially decays to zero as \( t \to \infty \).

3. Let \( u \) solve the following boundary value problem
\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{in} \quad \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, x_1 > t/2\}; \\
  u_t = 4u_{x_1} & \text{on} \quad \{x_1 = t/2\}.
\end{cases}
\]

Show that \( u(x,t) = 0 \) in \( \{|x| < R - t\} \cap \{x_1 > t/2\} \) when \( u(x,0) = u_t(x,0) = 0 \) in \( \{|x| < R\} \cap \{x_1 > 0\} \). Explain where the boundary condition on \( \{x_1 = t/2\} \) has been used.

4. Let \( V^k = \text{span}\{q_1, q_2, \ldots, q_k\}, q_i \neq 0 \in L^2(0,1), q_i(0) = q_i(1) = 0, \int_0^1 q_i(x)q_j(x)dx = \delta_{ij} \) where \( \delta_{ij} = 1 \) for \( i = j \) and \( \delta_{ij} = 0 \) for \( i \neq j \). Define \( A \in \mathbb{R}^{k \times k} \) with \( a_{ij} = \)
\[ \int_0^1 \frac{\partial q_i}{\partial x}(x) \frac{\partial q_j}{\partial x}(x) \, dx \] with eigenvalue decomposition \( A = V \hat{\Lambda} V^T \) where \( \hat{\Lambda} \) has diagonal entries \( \hat{\lambda}_i \) and \( V \) is orthogonal with entries \( v_{ij}, i, j = 1, 2, \ldots, k \). Show that \( r_i \in \mathcal{V}^{k\perp} = \{ f \in L^2(0, 1) \mid \int_0^1 f(x)q(x) \, dx = 0 \, \forall q \in \mathcal{V}^k \} \) with \( r_i(x) = -\frac{\partial^2 w_i}{\partial x^2}(x) - \hat{\lambda}_i w_i(x) \) and \( w_i(x) = \sum_j v_{ij} q_j(x) \).

5. Consider the PDE
\[ \frac{\partial^2 u}{\partial t^2}(x, t) = \frac{\partial^2 u}{\partial x^2}(x, t), \quad x \in (0, 1), \quad t > 0 \]
\[ u(x, 0) = (s - 1)x, \quad \frac{\partial u}{\partial t}(x, 0) = 0 \]
\[ \frac{\partial u}{\partial x}(0, t) = 0, \quad \frac{\partial u}{\partial x}(1, t) = 0. \]
for constant \( s \in \mathbb{R} \).

(a) Solve the PDE. Hint: solve in terms of the even extension \((u^e : \mathbb{R} \to \mathbb{R})\) of the initial data where
\[ u^e(x) = \begin{cases} (s - 1)\hat{x}, & \hat{x} \in [0, 1) \\ (s - 1)(2 - \hat{x}), & \hat{x} \in [1, 2) \end{cases} \]
with \( \hat{x} = 2(x/2 - \text{floor}(x/2)) \) for \( x \in \mathbb{R} \). floor(y) is the closest integer to y with floor(y) \( \leq y \).

(b) Define \( e(t) = \int_0^1 \left( \frac{\partial u}{\partial t}(x, t) \right)^2 + \left( \frac{\partial u}{\partial x}(x, t) \right)^2 \, dx \). Show that \( e(t) = (s - 1)^2 \).

6. Explain whether the ordinary differential equation:
\[ 5y'' + \left( \frac{y'}{x} \right)^2 + 4y^2 = 0 \]
has a unique smooth solution in a neighborhood of \( x = 0 \), when initial conditions \( y(0) = 1, y'(0) = 0 \) are applied.

Hint: Start by making the change of variables \( y(x) = 1 + cx^2 + v(x) \), where \( v(x) = o(x^2) \) as \( x \to 0 \), and \( c \) is a constant that needs to be determined.

7. Evolutionary rock-paper-scissors games are used to model interactions among bacteria. Consider three species of bacteria, with relative abundances \( R \) (rock), \( P \) (paper) and \( S \) (scissors) respectively. You may assume that \( P + R + S = 1 \). A \( R \)-type bacteria tends to out compete \( S \)-type bacteria, but is itself outcompeted by \( P \)-type bacteria.
The growth rate of the $R$-population is therefore proportional to the number of interactions each $R$-type bacteria has with $S$-types, minus the number of interactions with $P$ types. That is:

$$\dot{R} = R(S - P)$$

similarly:

$$\dot{S} = S(P - R)$$
$$\dot{P} = P(R - S)$$

(a) Describe all of the possible behaviors of the system if $R = 0$ at $t = 0$.

(b) Show that, if all three population types are present in the system at $t = 0$ (i.e. $R, P, S$ are all initially non-zero), then none of the types of bacteria will go extinct. That is, none of the variables $P, R$ or $S$ converges to 0, as $t \to \infty$.

8. The space $y > 0$ is filled with a non-Newtonian fluid, initially at rest. A plate at $y = 0$ is set into motion at time $t = 0$. The fluid velocity, $u(t, y)$ then obeys an equation:

$$\frac{\partial u}{\partial t} = -\frac{\partial \tau}{\partial y}, \quad t > 0, \; y > 0, \tag{1}$$

with boundary conditions:

$$u(t, 0) = 1, \quad \text{and} \quad u(t, +\infty) = 0 \tag{2}$$

and initial condition $u(0, y) = 0$ for $y > 0$. $\tau$ is assumed to obey a constitutive equation:

$$\tau = \left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

(a) Try to derive a similarity solution i.e. look for a solution of the form $u(t, y) = f(\eta)$ where $\eta = y/\delta(t)$ for some function $\delta(t)$, that you will need to determine), by applying only the boundary condition $u(t, 0) = 1$. Show that this similarity solution can not be compatible with the other boundary condition, or with the initial condition.

(b) To find a solution that is compatible with all boundary and initial conditions we modify the constitutive equation to:

$$\tau = \begin{cases} 
\left(\frac{\partial u}{\partial y}\right)^2 & \text{if } \frac{\partial u}{\partial y} < 0 \\
0 & \text{if } \frac{\partial u}{\partial y} \geq 0
\end{cases} \tag{4}$$

Derive a similarity solution that satisfies all of the initial and boundary conditions.

*Hint:* Start by assuming that the solution breaks down into two parts: $0 < y < Y(t)$, in which $\tau \neq 0$ and $y > Y(t)$ in which $\tau = 0$. Derive continuity conditions that must be applied at $y = Y(t)$. You need to solve for the function $Y(t)$, as well as for $f(\eta)$. 
