ADE Exam, Spring 2018

You have four hours to complete this exam. Start each question on a new sheet of paper, and write your UID on each answer sheet. Your name should not appear on any of the work that you submit.

1. Consider the following non-dimensionalized model for glycolysis:

\[
\begin{align*}
\dot{x} &= -x + ay + x^2y, \\
\dot{y} &= b - ay - x^2y,
\end{align*}
\]

where \(x \geq 0\) is the concentration of ADP, \(y \geq 0\) is the concentration of F6P, and \(a, b > 0\) are kinetic parameters. Determine the equilibrium points and their linear stability, and show that a periodic orbit exists if and only if \(a\) and \(b\) satisfy an appropriate condition (which you should determine). Draw the phase portrait in this case.

2. Consider the ordinary differential equation

\[
Ly \equiv -(p(x)y')' + q(x)y = \delta(x - \xi),
\]

where \(x \in (a,b)\) and the function \(p(x) \neq 0\) on \((a,b)\).

(a) Derive the Green’s function solution \(G(x,\xi)\). Your answer will contain the unknown functions \(p(x)\) and \(q(x)\).

(b) Give an alternative expression for \(G(x,\xi)\), in terms of an eigenfunction expansion, and show that the two formulas agree.

3. Consider the ordinary differentiation equation

\[
x^3 \frac{d^2y}{dx^2} + y = 0.
\]

(a) Show that the ODE has a regular singular point at \(x = \infty\) and determine its indicial exponents.

(b) The leading behavior of a particular solution to (3) is \(t(x) \sim x\) (as \(x \to \infty\)). By considering the largest terms in a singular series solution, determine the next-largest term in the expansion of \(y(x)\) for large positive \(x\).

4. We seek a solution of \(u : \Omega \times \mathbb{R} \to \mathbb{R}\) solves the partial differential equation:

\[
u_t = \Delta u - u||Du||
\]

where \(\Omega \subset \mathbb{R}^d\) is the interior of a connected compact set, and \(||\cdot||\) is the usual Euclidean norm. If smooth boundary conditions \((u(x, t) = f(x, t)\) on \(\partial \Omega)\) and initial conditions \((u(x, 0) = g(x))\) are specified, show that there is at most one \(C^{1,2}(\Omega \times \mathbb{R})\) solution of this PDE.
5. Consider entropy solutions, \( u(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \) of the flux-conservation equation:

\[
  u_t + (f(u))_x = 0
\]

with initial condition

\[
  u(x,0) = \begin{cases} 
    x, & \text{if } 0 < x < 1, \\
    0, & \text{otherwise}.
  \end{cases}
\]

and flux function \( f(u) = \frac{u^3}{3} \).

(a) Derive the Rankine-Hugoniot condition for the propagation of discontinuous solutions of this PDE.

(b) Find the long time solution of the PDE. You may assume that \( u \geq 0 \), so \( f(u) \) is convex, and also that at long times, the solution can be broken into three parts:

\[
  u(x,t) = \begin{cases} 
    0, & \text{if } x < 0, \\
    t^\alpha g \left( \frac{x}{t^\beta} \right), & \text{if } 0 < x < h(t), \\
    0, & \text{if } x > h(t).
  \end{cases}
\]

for some exponents \( \alpha \) and \( \beta \), and positive functions \( g \) and \( h \), all of which you should determine.

6. **Elastodynamics** is the study of how waves of vibration propagate through elastic bodies. Small deformations of an infinite elastic body can be modeled by a displacement vector field \( \mathbf{u}(x,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3 \). This displacement field must satisfy an equation:

\[
  \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times (\nabla \times \mathbf{u})
\]

where \( \rho \) (density) and \( \lambda, \mu \) (elastic moduli) are all positive constants.

(a) Suppose that we define fields \( \omega = \nabla \times \mathbf{u} \) and \( \epsilon = \nabla \cdot \mathbf{u} \). Show that these fields both satisfy wave equations and find the corresponding wave speeds in terms of the constants \( \rho, \lambda \) and \( \mu \).

(b) Assume that the displacement field within the solid is spherically symmetric; that is: \( \mathbf{u}(x,t) = u(r,t) \mathbf{e}_r \), where \( \mathbf{e}_r \) is a unit vector in the radial direction, and \( r^2 = x^2 + y^2 + z^2 \). In this case, show that \( \omega = 0 \).

(c) Assume a spherically symmetric displacement field, with initial conditions \( \epsilon(r,0) = \frac{\phi(r)}{r} \) and \( \epsilon_t(r,0) = 0 \). (You may assume that \( \phi \) is \( C^2 \), with \( \phi(0) = 0 \))

(i) Derive an expression for \( \epsilon(r,t) \).

(ii) Given the initial condition:

\[
  \epsilon(r,0) = \begin{cases} 
    1, & \text{if } r < 1, \\
    0, & \text{if } r > 1.
  \end{cases}
\]

calculate \( u(r,t) \) (you can assume that \( \lim_{r \to \infty} u = 0 \)). Hint: To relate \( u \) and \( \epsilon \), it is helpful to calculate \( \int_{B(0,r)} \epsilon \, dV \).
7. Let \( \Omega \subset \mathbb{R}^d \) be a bounded open set of smooth boundary \( \partial \Omega \). Recall the notation \( g \in C^1(\Omega) \) means there exists an open set \( O \) containing \( \Omega \) such that \( g \in C^1(O) \).

Let \( f_1, \cdots, f_d \in C^1(\bar{\Omega}) \) be such that
\[
\sum_{i=1}^d \frac{\partial f_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega.
\]

Suppose \( u \in C^2(\bar{\Omega}) \) and
\[
\triangle u + \sum_{i=1}^d f_i \frac{\partial u}{\partial x_i} - u^3 - u^5 = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega
\]
Show that \( u \) is identically zero on \( \Omega \).

8. Let \( \Phi \in C^3(\mathbb{R}^d) \) be such that \( \Phi \) and its first derivatives are bounded. We consider the Lagrangian
\[
L(x, v) = \frac{1}{2}|v|^2 - \Phi(x).
\]

Given \( 0 < T \leq \infty \) and \( x, y \in \mathbb{R}^d \), we define the minimal action
\[
C(x, y) = \inf_{\sigma} \left\{ \int_0^T L(\sigma(\tau), \dot{\sigma}(\tau)) d\tau \mid \sigma \in C^1([0, T]), \, \sigma : [0, T] \to \mathbb{R}^d, \, \sigma(0) = x, \, \sigma(T) = y \right\}. \tag{4}
\]

(a) Show that if \( \Phi \equiv 0 \) then \( \sigma_0(\tau) = (1 - \frac{\tau}{T}) x + \frac{\tau}{T} y \) is the unique path minimizing (4).

(b) Show that if \( \Phi \) is concave then (4) has at most one minimizer.