Algebra Qualifying Exam
Winter 2002

Everyone must do two problems in each of the four sections. If three problems of a section are tried, only two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

Groups
A1. Let \( G \) be a free abelian group of rank \( n \) for a positive integer \( n \) (therefore \( G \cong \mathbb{Z}^n \) as groups).
   (a) Prove for a given integer \( m > 1 \), there are only finitely many subgroups \( H \) of index \( m \) in \( G \);
   (b) Find a formula of the number of subgroups of \( G \) of index 3. Justify your answer.
A2. Prove or disprove: there exists a finite abelian group \( G \) whose automorphism group has order 3.
A3. Let \( S \) and \( G \) be \( p \)-groups (with \( G \neq \{e\} \)), and assume that \( S \) acts on \( G \) by automorphisms. Show that the fixed subgroup \( G^S = \{g \in G| s(g) = g \text{ for all } s \in S\} \) is non-trivial (i.e., is not the trivial subgroup \( \{e\} \)).

Rings
B1. Let \( F \) be a field and \( A \) be a commutative \( F \)-algebra. Suppose \( A \) is of finite dimension as a vector space of \( F \).
   (a) Prove all prime ideals of \( A \) are maximal. Hint: consider maps \( R/P \to R/P \) (\( P \) prime) of the form \( x \to ax \) with \( a \) in \( R \).
   (b) Prove that there are only finitely many maximal ideals of \( A \).
B2. Let \( A = M_n(F) \) be the ring of \( n \times n \) matrices with entries in an infinite field \( F \) for \( n > 1 \). Prove the following facts:
   (a) There are only 2 two-sided ideals of \( A \);
   (b) There are infinitely many maximal left ideals of \( A \). Hint: show that \( Ax = Ay \) (\( x, y \in A \)) if and only if \( \text{Ker}(x) = \text{Ker}(y) \).
B3. Let \( \mathbb{F}_2 \) be the field with 2 elements and \( A = \mathbb{F}_2[T, \frac{1}{T}] \) for an indeterminate \( T \). Prove the following facts:
   (a) The group of units in \( A \) is generated by \( T \).
   (b) There are infinitely many distinct ring endomorphisms of \( A \).
   (c) The ring automorphism group \( \text{Aut}(A) \) is of order 2.
Fields

C1. The discriminant of the special cubic polynomial \( f(x) = x^3 + ax + b \) is given by \(-4a^3 - 27b^2\). Determine the Galois group of the splitting field of \( x^3 - x + 1 \) over
(a) \( \mathbb{F}_3 \), the field with 3 elements.
(b) \( \mathbb{F}_5 \), the field with 5 elements.
(c) \( \mathbb{Q} \), the rational numbers.

C2. A field extension \( K/\mathbb{Q} \) is called biquadratic if it has degree 4 and if \( K = \mathbb{Q}(\sqrt{a}, \sqrt{b}) \) for some \( a, b \in \mathbb{Q} \).
(a) Show that a biquadratic extension is normal with Galois group \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and list all sub-extensions.
(b) Prove that if \( K/\mathbb{Q} \) is a normal extension of degree 4 with \( \text{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) then \( K/\mathbb{Q} \) is biquadratic.

C3. Let \( K \) be a finite extension of the field \( F \) with no proper intermediate subfields.
(a) If \( K/F \) is normal, show that the degree \([K; F] \) is a prime.
(b) Give an example to show that \([K; F] \) need not be prime if \( K/F \) is not normal, and justify your answer.

Linear Algebra

D1. Let \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) where \( I \) is the \( n \times n \) identity matrix. Suppose that \( S \) is a \( 2n \times 2n \) symplectic matrix, meaning that \( S \) is real and satisfies \( ^tSJS = J \), where \( ^tS \) is the transpose of \( S \).
(a) Show that \( ^tS \) is symplectic.
(b) Show that \( S \) is similar to \( S^{-1} \).
(c) It is always true that det \( S = 1 \). Prove this in case \( n = 1 \).

D2. Suppose that \( A \) is a linear operator on the vector space \( \mathbb{C}^n \) and that \( v \in \mathbb{C}^n \) satisfies \((A - aI)^2v = 0 \) for some \( a \in \mathbb{C} \), so that \( v \) is a generalized eigenvector of \( A \) with eigenvalue \( a \). Suppose that \(|a| < 1 \). Show that
\[
\|A^m v\| \to 0
\]
as \( m \to \infty \), where \( \|\cdot\| \) is the Euclidean norm on \( \mathbb{C}^n \).

D3. Let the \( n \times n \) matrix \( A \) be defined over the field \( F \). Suppose that \( A \) has finite order:
\[
A^m = I
\]
for some positive integer \( m \).
(a) If the characteristic of \( F \) is 0, show that \( A \) may be diagonalized over \( F \).
(b) Show that the conclusion of (a) is not true for an arbitrary field \( F \).