Algebra Qualifying Exam (Spring 2004)

Test Instructions: Everyone must do two problems in each of the four sections. If three problems of a section are tried, only the two problems of highest score count (the lowest score is ignored). On multiple part problems, do as many parts as you can; however, not all parts count equally.

GROUP THEORY

PROBLEM 1.

A group $G$ is said to act transitively on a set $S$ if for any element $s \in S$, then

$$S = Gs.$$  

Suppose $G$ is finite and that $G$ acts transitively on $S$. Let $f(g)$ be the number of elements of $S$ fixed by the action of $g \in G$ on $S$. Prove

$$|G| = \sum_{g \in G} f(g).$$

PROBLEM 2.

Classify all groups of order $2 \cdot 7 \cdot 11$.

PROBLEM 3.

Let $G$ be a finite group and $H$ a subgroup of $G$. Let $n = (G : H)$ be the index of $H$ in $G$.

(a) Show that

$$(G : \bigcap_{x \in G} xHx^{-1})$$

is a factor of $n!$.

(b) Suppose that the index $(G : H)$ is the minimal prime factor of the order of $G$. Show $H$ is a normal subgroup.
RING THEORY

PROBLEM 1.
Let \( R \) be a commutative noetherian ring with unity 1 and \( f : R \to R \) a surjective ring homomorphism, i.e. \( f(R) = R \). Show \( f \) is an isomorphism.

PROBLEM 2.
Let \( R \) be the ring \( \mathbb{Q}[x] \) and let \( M \) be the submodule of \( R^2 \) generated by the elements \((1 - 2x, -x^2)\) and \((1 - x, x - x^2)\). According to the theory of modules over principal ideal domains, \( R^2/M \) is a direct sum of cyclic \( R \) modules of the form \( R/P(x) \) for monic polynomials \( P(x) \). Find such a direct sum decomposition explicitly in this case.

PROBLEM 3.
Suppose we are given a collection of polynomials in \( r \) variables with rational coefficients:
\[
 f_1, \ldots, f_N \in \mathbb{Q}[T_1, \ldots, T_r].
\]
We define the complex algebraic set \( V_C \subset \mathbb{C}^r \) by
\[
 V_C = \{(a_1, \ldots, a_r) \mid f_i(a_1, \ldots, a_r) = 0 \text{ for all } i \text{ from } 1 \text{ to } N\}.
\]
Suppose \( V_C \) is not empty. Show that there is a finite extension \( K \) of \( \mathbb{Q} \) and a point \( (a_1, \ldots, a_r) \in V_C \) with all \( a_k \in K \).
LINEAR ALGEBRA

PROBLEM 1.
(a) For which $z \in \mathbb{C}$ is
\[
\begin{pmatrix}
1 & 2z \\
z - 1 & 1
\end{pmatrix}
\]
not similar over $\mathbb{C}$ to a diagonal matrix? Justify your answer.

(b) Let $J_n$ be the $n \times n$ matrix each of whose entries is 1. Determine those $n \in \mathbb{Z}^+$ for which $J_n$ is diagonalizable over $\mathbb{C}$ and give a diagonal matrix that is similar to $J_n$ for such $n$.

PROBLEM 2.
Find an explicit formula for the determinant of a $3 \times 3$ complex matrix $A$ as a polynomial in the traces $t_n = \text{Tr}(A^n)$ for $n = 1, 2, \ldots$.

PROBLEM 3.
Let $V$ be a vector space over $\mathbb{C}$ of dimension $d > 0$. Suppose that $A, B, C$ are linear operators on $V$ such that

\[AB - BA = C.\]

Suppose also that $C$ commutes with both $A$ and $B$. If $V$ has no proper non-zero subspace that is left stable under all three operators, show that $d = 1$. 

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FIELD THEORY

PROBLEM 1.
Let $K$ be a finite extension of $\mathbb{Q}$ obtained by adjoining to $\mathbb{Q}$ a root of $f(x) = x^6 + 3$.

(a) Show that $K$ contains a primitive 6-th root of unity.

(b) Show that $K$ is a Galois extension of $\mathbb{Q}$.

(c) Determine the number of fields $F$ of degree 3 over $\mathbb{Q}$ with $F \subseteq K$.

PROBLEM 2.
Suppose that $f(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree $d > 1$ with $d$ roots $x_1, \ldots, x_d$ in $\mathbb{C}$. If $x_2 = ax_1$ for $a \in \mathbb{Q}$ different from $-1$, prove that $f(x)$ is reducible.

PROBLEM 3.
Let $K$ be a field and $L$ a finite extension of $K$. Consider the set $A$ of all elements $x \in L$ with the property that $K[x]$ is a Galois extension of $K$ with an abelian Galois group $\text{Gal}(K[x]/K)$. Show that $A$ is a subfield of $L$ containing $K$.