Algebra Qualifying Exam
Spring 2008

Test Instructions: Each problem is worth 20 points. You should do two problems in each section. In particular, do not submit more than two problems in each section. In each problem, you must explain clearly your reasoning and any theorems that you quote. Whether you pass or fail depends on your performance in each section, not only on the total score.

GROUPS

PROBLEM G1
Let \( p \) be a prime number. Show that a subgroup \( G \) of \( S_p \) which contains an element of order \( p \) and which contains a transposition must be the whole of \( S_p \).

PROBLEM G2
Let \( G = D_{2n} \) be the dihedral group of order \( 2n \) where \( n \geq 3 \). Prove that \( \text{Aut}(G) \) is isomorphic to the group of \( 2 \times 2 \) matrices of the form

\[
H = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} : \alpha \in (\mathbb{Z}/n)^*, \beta \in \mathbb{Z}/n \right\}.
\]

PROBLEM G3
Let \( K \) be a normal subgroup of a finite group \( G \). Let \( p \) be a prime. Let \( N_p \) be the number of \( p \)-Sylow subgroups (\( p \)-SSG) of \( G \) and \( N'_p \) the number in \( K \).

(a) Show that \( N_p = |G : N_G(P)| \) where \( P \) is any \( p \)-SSG of \( G \) and \( N_G(P) \) is the normalizer of \( P \) in \( G \).

(b) Prove that \( N'_p \) divides \( N_p \).
RINGS

PROBLEM R1
Let $D$ be an associative ring with unit having no zero divisors. Assume that the center of $D$ contains a field $k$ such that $\dim_k(D) < \infty$. Prove that $D$ is a division algebra (i.e., every non-zero element is invertible).

PROBLEM R2
Let $G$ be a finite group of order $|G| > 1$. The rational group ring $\mathbb{Q}[G]$ of $G$ is the $\mathbb{Q}$-algebra consisting of all finite linear combinations

$$\sum_{g \in G} a_g g$$

where $a_g \in \mathbb{Q}$. Multiplication in $\mathbb{Q}[G]$ is defined by extending the group multiplication linearly.

(a) Show that $\mathbb{Q}[G]$ has a non-trivial idempotent: $\exists a \in \mathbb{Q}[G], a \neq 0,1$ with $a^2 = a$.

Hint: reduce to the case of a cyclic group $G = \langle x : x^n = 1 \rangle$.

(b) Show that $\mathbb{Q}[G]$ contains an invertible element $u$ that is non-trivial, that is, not of the form $u = ag$ where $a \in \mathbb{Q}$ and $g \in G$.

(FYI: The Kadison-Kaplansky Conjecture claims that for $G$ without torsion, the group algebra $\mathbb{Q}[G]$ contains no non-trivial idempotent. This conjecture is open in general.)

PROBLEM R3
Let $R$ be a Noetherian ring and $I$ any ideal of $R$. Prove that there exist prime ideals $P_1, \ldots, P_m$ of $R$ such that

$$P_1P_2 \cdots P_m \subset I$$

Hint: Show that if $J$ is any non-prime ideal, then there exist $a, b \notin J$ such that $(J + a)(J + b) \subset J$. Then use the Noetherian property.
FIELDS

PROBLEM F1
Consider the polynomial $P(X) = X^5 - 4X + 2$ in $\mathbb{Q}[X]$.

(a) Show that $P$ is irreducible and has 3 real roots and 2 complex ones.

(b) Show that the Galois group of $P$ is $S_5$.

PROBLEM F2
Let $\zeta_n = \exp(2\pi i/n)$ be a primitive $n$th root of unity. Let $F_n = \mathbb{Q}(\zeta_n)$. Set

$$d_n = [F_n : \mathbb{Q}]$$

(a) Let $n = 6$. Find an irreducible polynomial of degree $d_6$ in $\mathbb{Q}[x]$ whose roots generate $F_6$.

(b) Let $n = 12$. Find an irreducible polynomial of degree $d_{12}$ in $\mathbb{Q}[x]$ whose roots generate $F_{12}$.

PROBLEM F3
Let $E/F$ be a finite, separable extension of fields. Prove that there exists $\alpha \in E$ such that $E = F(\alpha)$. State clearly any theorems you use in the proof.
LINEAR ALGEBRA

PROBLEM LA1
Let $V$ and $W$ be vector spaces over a field $F$ and let $V^*$ be the dual space of $V$. Let $\text{Hom}(V,W)$ be the space of linear maps from $V$ to $W$. There exists a natural linear map

$$T : V^* \otimes_F W \rightarrow \text{Hom}(V,W)$$

defined by $T(f \otimes w)(v) = f(v) \cdot w$. Show that $V$ is finite dimensional if and only if $T$ is an isomorphism for all $W$.

PROBLEM LA2
Let $M_4(\mathbb{Q})$ be the ring of all $4 \times 4$ matrices with coefficients in $\mathbb{Q}$. Find a set of representatives for the conjugacy classes of elements $X \in M_4(\mathbb{Q})$ satisfying the equation $X^4 = 2X^2$.

PROBLEM LA3
Let $V$ be a finite dimensional $F$-vector space and $T : V \rightarrow V$ a linear endomorphism. Show that there exists a decomposition

$$V = V_1 \oplus V_2$$

with the properties:

1. $T(V_i) \subset V_i$ for $i = 1, 2$
2. $T$ is an isomorphism on $V_1$
3. $T$ is nilpotent on $V_2$.

Hint: Consider the sequences of subspaces $\text{Im}(T) \supset \text{Im}(T^2) \supset \cdots$ and that $\text{Ker}(T) \subset \text{Ker}(T^2) \subset \cdots$. 