ALGEBRA QUALIFYING EXAM
FALL 2011

Do all the following 10 problems (see reverse). Good luck!

Problem 1. For a finite field $\mathbb{F}$, prove that the order of the group $\text{SL}_2(\mathbb{F})$, of $2 \times 2$ matrices with determinant 1, is divisible by 6.

Problem 2. Let $G$ be a non-trivial finite group and $p$ a prime. If every subgroup $H \neq G$ has index divisible by $p$, prove that the center of $G$ has order divisible by $p$.

Problem 3. Let $R$ be a local UFD of Krull dimension 2 (meaning that the maximal integer $m$ for which there exist strict inclusions of prime ideals $p_0 \subseteq p_1 \subseteq \cdots \subseteq p_m$ in $R$ is exactly 2). Let $\pi \in R$ be neither zero nor a unit. Prove that $R[\frac{1}{\pi}]$ is a PID.

Problem 4. Let $p$ be a prime. Prove that the nilradical of the ring $\mathbb{F}_p[X] \otimes_{\mathbb{F}_p(X)} \mathbb{F}_p[X]$ is a principal ideal.

Problem 5. Let $F$ be a finite field and $\overline{F}$ be an algebraic closure of $F$. Let $K$ be a subfield of $\overline{F}$ generated by all roots of unity over $F$. Show that any simple $K$-algebra of finite dimension over $K$ is isomorphic to the matrix algebra $M_n(K)$ for a positive integer $n$.

Problem 6. Let $R$ be a commutative ring and let $M$ be a finitely generated $R$-module. Let $f : M \to M$ be $R$-linear such that $f \otimes \text{id} : M \otimes_R R[T] \to M \otimes_R R[T]$ is surjective. Prove that $f$ is an isomorphism.

Problem 7. Let $C$ be the category of semi-symplectic topological quantum paramonoids of Rice-Paddy type, satisfying the Mussolini-Rostropovich equations at infinity. Let $X, Y$ be objects of $C$ such that the functors $\text{Mor}_C(X, -)$ and $\text{Mor}_C(Y, -)$ are isomorphic, as covariant functors from $C$ to sets. Show that $X$ and $Y$ are isomorphic in $C$.

Problem 8. Let $\Gamma$ be the Galois group of the polynomial $X^5 - 9X + 3$ over $\mathbb{Q}$. Determine $\Gamma$. [Hint: Show that $\Gamma$ contains an element of order 5 and that $\Gamma$ contains a transposition, in a sense to be made precise.]
Problem 9. (We denote by $\mathbb{F}G$ the group algebra of $G$.)

(a) Is there a group $G$ with $\mathbb{C}G$ isomorphic to $\mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C})$?

(b) Is there a group $G$ with $\mathbb{Q}G$ isomorphic to $\mathbb{Q} \times \mathbb{Q} \times M_3(\mathbb{Q})$?

Problem 10. Let $K/k$ be an extension of finite fields. Show that the norm $N_{K/k} : K \rightarrow k$ is surjective.