(1) Let $p$ be a prime integer and let $G$ be a (finite) $p$-group. Write $C$ for the subgroup of central elements $x \in G$ satisfying $x^p = 1$. Let $N$ be a normal subgroup of $G$ such that $N \cap C = \{1\}$. Prove that $N = \{1\}$.

(2) Let $A$ be an $n \times n$ matrix over a field $F$ having only one invariant factor. Prove that every $n \times n$ matrix over $F$ that commutes with $A$ is a polynomial in $A$ with coefficients in $F$.

(3) Let $F$ be a field and let $n$ be a positive integer such that $F$ has no nontrivial field extensions of degree less than $n$. Let $L = F(x)$ be a field extension with $x^n \in F$. Prove that every element in $L$ is a product of elements of the form $ax + b$ with $a, b \in F$.

(4) Let $F$ be the functor from the category of rings to the category of sets taking a ring $R$ to the set $\{x^2 \mid x \in R\}$. Determine whether $F$ is representable.

(5) Let $G$ and $H$ be finite groups and let $V$ and $W$ be irreducible (over $\mathbb{C}$) $G$- and $H$-modules respectively. Prove that the $G \times H$-module $V \otimes W$ is also irreducible.

(6) Let $D_n$ be a dihedral group of order $2n > 4$; so, it contains a cyclic subgroup $C$ of order $n$ on which $\sigma \in D_n$ outside $C$ acts as $\sigma c \sigma^{-1} = c^{-1}$ for all $c \in C$. When is the cyclic subgroup $C$ with the above property unique? Determine all $n$ for which $D_n$ has unique cyclic subgroup $C$ and justify your answer.

(7) Let $D$ be a central simple division algebra of dimension 4 over a field $F$. If a quadratic extension $K/F$ can be isomorphically embedded into $D$ as $F$-algebras, prove that $D \otimes_F K$ is isomorphic to the $2 \times 2$ matrix algebra $M_2(K)$ as $K$-algebras.

(8) How many monic irreducible polynomials over $\mathbb{F}_p$ of prime degree $l$ are there? Here $\mathbb{F}_p$ is the field of $p$ elements for a prime number $p$. Justify your answer.

(9) Consider a covariant functor $F : R \mapsto R^\times$ from the category of commutative rings with a multiplicative identity into the category of sets. Let $G = \text{Aut}_{\text{functors}}(F)$ be made up of natural transformations $I : F \to F$ having an inverse $J : F \to F \in G$ such that $I \circ J = J \circ I$ is the identity natural transformation. Prove first that $F$ is representable by a ring, that $G$ is a finite set, and find the order of the group $G$ with justification.

(10) For a finite field $F$ of order $q$, consider the polynomial ring $R = \mathbb{F}[x]$, and let $L$ be a free $R$-module of rank 2. Give the number of $R$-submodules $M$ such that $xL \subseteq M \subseteq L$, and justify your answer.