Do the following ten problems.

1. How many groups are there up to isomorphism of order \(pq\) where \(p > q\) are prime integers?

2. Show that there are up to isomorphism exactly two nonabelian groups \(G\) of order 8. Prove that each of them has an irreducible complex representation of dimension 2.

3. For a positive integer \(n\), let \(\Phi_n(X)\) be the \(n\)th cyclotomic polynomial. If \(a\) is an integer and \(p\) a prime not dividing \(n\), such that \(p\) divides \(\Phi_n(a)\), show that the order of \(a \mod p\) is \(n\). Using this prove that there are infinitely many primes \(p\) such that \(p\) is 1 modulo \(n\).

4. Given a field \(K\) of characteristic \(p\), when is an \(\alpha\) that is algebraic over \(K\) said to be separable? Show that if \(\alpha\) is algebraic over \(K\), then \(\alpha\) is separable if and only if \(K(\alpha) = K(\alpha^{p^n})\) for all positive integers \(n\).

5. Let \(G\) be a finite group which has the property that for any element \(g \in G\) of order \(n\), and an integer \(r\) prime to \(n\), the elements \(g\) and \(g^r\) lie in the same conjugacy class. Then show that the character of every representation of \(G\) takes values in the rational numbers \(\mathbb{Q}\) (in fact even the integers \(\mathbb{Z}\)). (Hint: Use Galois theory.)

6. Let \(I\) be an ideal of a commutative ring and \(a \in R\). Suppose the ideals \(I + Ra\) and \((I : a) := \{x \in R \mid ax \in I\}\) are finitely generated. Prove that \(I\) is also finitely generated.

7. Give an example of a 10 \times 10 matrix over \(\mathbb{R}\) with minimal polynomial \((X + 1)^2(X^4 + 1)\) which is not similar to a matrix with rational coefficients.

8. Suppose that \(E/F\) is an algebraic extension of fields such that every nonconstant polynomial in \(F[X]\) has at least one root in \(E\). Show that \(E\) is algebraically closed.

9. Prove that is \(ab = 1\) in a semisimple ring, then \(ba = 1\).
10. Let $A$ be the functor from the category of groups to the category of (unital) rings taking a group $G$ to the group ring $\mathbb{Z}[G]$ of all finite formal sums $\sum_{g \in G} a_g g$ with $a_g \in \mathbb{Z}$ (the product in $\mathbb{Z}[G]$ is induced by the group operation in $G$). Prove that $A$ has a right adjoint functor.