1: Let $G$ be a free abelian group of rank $r$, so $G$ is isomorphic to $\mathbb{Z}^r$ as groups. Show that $G$ has only finitely many subgroups of a given finite index $n$.

2: Assume that $L$ is a Galois extension of the field of rational numbers $\mathbb{Q}$ and that $K \subset L$ is the subfield generated by all roots of unity in $L$. Suppose that $L = \mathbb{Q}[a]$, where $a^n \in \mathbb{Q}$ for some positive integer $n$. Show that the Galois group $\text{Gal}(L/K)$ is cyclic.

3: Let $K \subset L$ be an algebraic extension of fields. An element $a$ of $L$ is called abelian if $K[a]$ is a Galois extension of $K$ with abelian Galois group $\text{Gal}(K[a]/K)$. Show that the set of abelian elements of $L$ is a subfield of $L$ containing $K$.

4: Let $\mathbb{F}_2$ be the field with 2 elements and let $R = \mathbb{F}_2[x]$. List, up to isomorphism, all $R$–modules with 8 elements.

5: Let $R$ be a commutative local ring, so $R$ has a unique maximal ideal $M$.
   a) Show that if $x \in M$ then $1 - x$ is invertible.
   b) Show that if $R$ is Noetherian and if $I$ is an ideal such that $I^2 = I$ then $I = 0$.

6: Let $D$ be a division ring of characteristic 0. Assume that $D$ has dimension 2 as a $\mathbb{Q}$-vector space. Show that $D$ is commutative.

7: Let $F = \mathbb{F}_2$ be the field with two elements. Show that there is a ring homomorphism $F[GL_2(F)] \to M_2(F)$ that sends the element $g$ in the group ring to the matrix $g \in M_2(F)$. Show that this homomorphism is surjective. Let $K$ be the kernel; since it is a left ideal, it is a (left) $GL_2(F)$-module. Is this module indecomposable? (Reminder: a module is indecomposable if it is not the direct sum of two proper submodules.) Describe the simple modules in its composition series.

8: Let $\mathbb{C}$ and $\mathbb{D}$ be additive categories, and let $\Phi : \mathbb{C} \to \mathbb{D}$ be a functor. Show: If $\Phi$ has a right adjoint, then $\Phi$ commutes with direct sums and for any two objects $x$ and $y$ in $\mathbb{C}$, the map $\Phi_{x,y} : \text{Hom}_\mathbb{C}(x, y) \to \text{Hom}_\mathbb{D}(x, y)$ is a homomorphism.

9: Let $D$ be an associative ring without zero divisors, and assume the center of $D$ is a field over which $D$ is a finite-dimensional vector space. Prove that $D$ is a division algebra.

10: Let $G$ be a finite group of order $n$ and $\rho : G \to GL(V)$ a complex representation of $G$ of dimension $n$. Show that $\rho$ cannot be irreducible.