Please do the following ten problems. Write your UID number ONLY, not your name.

(1) Let $\alpha \in \mathbb{C}$. Suppose that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is finite and prime to $n!$ for an integer $n > 1$. Show that $\mathbb{Q}(\alpha^n) = \mathbb{Q}(\alpha)$.

(2) Let $\zeta^9 = 1$ and $\zeta^3 \neq 1$ with $\zeta \in \mathbb{C}$.
   (a) Show that $\sqrt[3]{3} \notin \mathbb{Q}(\zeta)$.
   (b) If $\alpha^3 = 3$, show that $\alpha$ is not a cube in $\mathbb{Q}(\zeta, \alpha)$.

(3) Let $\mathbb{Z}^n$ ($n > 1$) be made of column vectors with integer coefficients. Prove that for every non-zero left ideal $I$ of $\mathbb{M}_n(\mathbb{Z})$, $IZ^n$ (the subgroup generated by products $\alpha v$ with $\alpha \in I$ and $v \in \mathbb{Z}^n$) has finite index in $\mathbb{Z}^n$.

(4) Let $p$ be a prime number, and let $D$ be a central simple division algebra of dimension $p^2$ over a field $k$. Pick $\alpha \in D$ not in the center and write $K$ for the subfield of $D$ generated by $\alpha$. Prove that $D \otimes_k K \cong \mathbb{M}_p(K)$ (the $p \times p$ matrix algebra with entries in $K$).

(5) Let $C$ be a category. A morphism $f : A \to B$ in $C$ is called an epimorphism if for any two morphisms $g, h : B \to X$ in $C$, $g \circ f = h \circ f$ implies $g = h$. Let $\text{ALG}$ be the category of $\mathbb{Z}$-algebras, and let $\text{MOD}$ be the category of $\mathbb{Z}$-modules.
   (a) Prove that in $\text{MOD}$, $f : M \to N$ is an epimorphism if and only if $f$ is a surjection.
   (b) In $\text{ALG}$, does the equivalence of epimorphism and surjection hold? If yes, prove the equivalence, and if no, give a counterexample (as simple as possible).

(6) Let $G$ be a group with a normal subgroup $N = \langle y, z \rangle$ isomorphic to $(\mathbb{Z}/2)^2$. Suppose that $G$ has a subgroup $Q = \langle x \rangle$ isomorphic to the cyclic group $\mathbb{Z}/3$ such that the composition $Q \subset G \to G/N$ is an isomorphism. Finally, suppose that $\sigma x \tau = z$ and $\tau x \sigma^{-1} = \zeta$. Compute the character table of $G$.

(7) Let $B$ be a commutative noetherian ring, and let $A$ be a noetherian subring of $B$. Let $I$ be the nilradical of $B$. If $B/I$ is finitely generated as an $A$-module, show that $B$ is finitely generated as an $A$-module.

(8) Let $F$ be a field that contains the real numbers $\mathbb{R}$ as a subfield. Show that the tensor product $F \otimes_{\mathbb{R}} \mathbb{C}$ is either a field or isomorphic to the product of two copies of $F$, $F \times F$.

(9) Show that there is no simple group of order 616.
(10) By one definition, a Dedekind domain is a commutative noetherian integral domain $R$, integrally closed in its fraction field, such that $R$ is not a field and every nonzero prime ideal in $R$ is maximal. Let $R$ be a Dedekind domain, and let $S$ be a multiplicatively closed subset of $R$. Show that the localization $R[S^{-1}]$ is either the zero ring, a field, or a Dedekind domain.