ANALYSIS QUALIFYING EXAM FALL 2007

Instructions: Do 10 of the 13 problems, at least three from problems 9-13. All problems are worth the same amount.

1 Let $f: \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$, periodic of period $2\pi$, and satisfying

$$\int_{-\pi}^{+\pi} f(t) \, dt = 0.$$  

Prove that $\int_{-\pi}^{+\pi} |f(t)|^2 \, dt \leq \int_{-\pi}^{+\pi} |f'(t)|^2 \, dt$. Find all such functions for which equality holds.

2 Is there a closed uncountable subset of $\mathbb{R}$ which contains no rational numbers? Prove your answer.

3(a) Prove that a complete normed vector space (a Banach space) is either finite-dimensional or has uncountable dimension in the vector space sense, i.e., it is not generated as finite linear combinations of elements of some countable subset.

(b) Use part (a) to give an example of a vector space that cannot be given the structure of a Banach space, that is, it is not complete in any norm.

4 Prove that not every subset of $[0,1]$ is Lebesgue measurable.

5 (a) Prove that if $a_n$ is a decreasing sequence of positive numbers converging to 0, then the series $\sum (-1)^n a_n$ is convergent.

(b) Prove that if $f:(0, \infty) \to (0, \infty)$ is decreasing and $\lim_{x \to \infty} f(t) = 0$, then the improper integral $\int_{0}^{\infty} f(x) \cos x \, dx$ is convergent.

6 Derive, proving the validity carefully, some series or product expansion (your choice) that yields a method of calculating $\pi$.

7 Suppose $H$ is a separable Hilbert space.

(a) Prove: If $T:H \to H$ is a linear mapping such that $\|I-T\| < 1$, where $I$ is the identity map of $H$ to itself, then $T$ is invertible.

(b) Suppose $e_n, n=1,2,3,\ldots$ is a complete, orthonormal set in $H$ (a Hilbert space basis).

Suppose also that $f_n, n=1,2,3,\ldots$ is an orthonormal set in $H$ such that $\sum \|e_n - f_n\|^2 < 1$.

Prove that $f_n$ is a complete orthonormal set in $H$.

8 Let $f:R \to R$ be a continuous function that is periodic with period $2\pi$. Show that if all the Fourier coefficients of $f$ (for $[-\pi, +\pi]$) are 0, then $f$ is identically 0.
9 Suppose that \( h: D\setminus\{(0,0)\} \to \mathbb{R} \) is a harmonic function, where \( D \) is the unit disc in the complex plane.

(a) Show that there is exactly one real number \( A \) such that \( h(z) - A \ln |z| \) is the real part of a holomorphic function on \( D\setminus\{(0,0)\} \). (Hint: Consider a candidate for a harmonic conjugate constructed by contour integration.)

(b) Use part (a) to show that if \( h \) is bounded then \( h \) extends to be a harmonic function on \( D \).

10 Let \( U \) be a bounded connected open set in \( \mathbb{C} \). Prove that if \( K \) is a compact subset of \( U \), then there is a constant \( C_K \) such that for every point \( z \) in \( K \) and every holomorphic \( L^2 \) function \( f \) on \( U \),

\[
|f(z)| \leq C_K \left( \int_U |f|^2 \right)^{1/2}.
\]

11 Let \( U \) be a bounded connected open set in \( \mathbb{C} \). Suppose that 0 belongs to \( U \) and that \( F:U \to U \) is a holomorphic function with \( F(0)=0 \) and with \( F'(0)=1 \). Prove that \( F \) is the identity map. [Suggestion: Consider the power series of \( F \) composed with itself many times].

12 Find a conformal map to the unit disc of the half disc \( \{ z : |z| < 1, \Re z > 0 \} \). You may write your answer as a composition of simpler conformal maps.

13 Suppose \( R \) is a positive number and \( U=\{ z : |z| < R \} \)

(a) Show that for each holomorphic function \( f \) on \( U \), there is a power series \( \sum_{n=0}^{\infty} a_n z^n \) which converges at each point \( z \) in \( U \) to \( f(z) \).

(b) Show that there is only one such power series.