Instructions: Attempt ten of the thirteen questions, including at least three from Q9-13. Each question is worth 10 points.

Q1. Let $S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$ denote the unit circle. Show that there exists a measurable function $f : S^1 \to S^1$ whose Fourier coefficients $f(n) := \frac{1}{2\pi} \int_0^{2\pi} f(e^{2\pi i \theta}) e^{-2\pi i n \theta} \, d\theta$ are non-zero for every integer $n \in \mathbb{Z}$. (Hint: use the Baire category theorem.)

Q2. Let $R/Z$ be the unit circle with the usual Lebesgue measure. For each $n = 1, 2, 3, \ldots$, let $K_n : R/Z \to R^+$ be a non-negative integrable function such that $\int_{R/Z} K_n(t) \, dt = 1$ and $\lim_{n \to \infty} \int_{|t| \leq 1/2} K_n(t) \, dt = 0$ for every $0 < \varepsilon < 1/2$, where we identify $R/Z$ with $(-1/2, 1/2]$ in the usual manner. (Such a sequence of $K_n$ are known as approximations to the identity.) Let $f : R/Z \to R$ be continuous, and define the convolutions $f \ast K_n : R/Z \to R$ by

$$f \ast K_n(x) := \int_{R/Z} f(x - t)K_n(t) \, dt.$$  

Show that $f \ast K_n$ converges uniformly to $f$.

Q3. Let $X$ be a compact metric space.

- (a) Show that $X$ is separable (i.e. it has a countable dense subset).
- (b) Show that $X$ is second countable (i.e. there exists a countable base for the topology).
- (c) Show that $C(X)$ (the space of continuous functions $f : X \to R$ with the uniform topology) is separable. (Hint: use part (b), Urysohn's lemma and the Stone-Weierstrass theorem.)
Q4. Let \( f, g \in L^2(\mathbb{R}) \) be two square-integrable functions on \( \mathbb{R} \) (with the usual Lebesgue measure). Show that the convolution

\[
 f \ast g(x) := \int_{\mathbb{R}} f(y)g(x - y) \, dy
\]

of \( f \) and \( g \) is a bounded continuous function on \( \mathbb{R} \).

Q5. Let \( H \) be a Hilbert space, and let \( T : H \to H \) be a bounded linear operator on \( H \).

- Show that if the operator norm \( \|T\| \) of \( T \) is strictly less than 1, then the operator \( 1 - T \) is invertible.

- Let \( \sigma(T) \) denote the set of all complex numbers \( z \) such that \( T - zI \) is not invertible. (This set is known as the spectrum of \( T \).) Show that \( \sigma(T) \) is a compact subset of \( \mathbb{C} \).

Q6. Let \( \mu_n \) be a sequence of Borel probability measures on \([0,1]\), thus each \( \mu_n \) is a non-negative finite measure on the Borel \( \sigma \)-algebra of \([0,1]\) (the \( \sigma \)-algebra generated by the open sets in \([0,1]\)) with \( \mu_n([0,1]) = 1 \). Show that there exists a subsequence \( \mu_{n_j} \), as well as another Borel probability measure \( \mu \), such that \( \lim_{j \to \infty} \int_{[0,1]} f(x) \, d\mu_{n_j}(x) = \int_{[0,1]} f(x) \, d\mu(x) \) for all continuous functions \( f : [0,1] \to \mathbb{R} \). (Hint: use the Riesz representation theorem and Q3.)

Q7. Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be a bounded smooth function, and suppose that the Laplacian \( \Delta u(x, y) := \frac{\partial^2 u}{\partial x^2}(x, y) + \frac{\partial^2 u}{\partial y^2}(x, y) \) of \( u \) is rotationally symmetric, which means that \( \Delta u(R_\theta(x, y)) = \Delta u(x, y) \) for any rotation \( R_\theta : (x, y) \mapsto (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) \). Show that \( u \) is also rotationally symmetric. (Hint: You may use without proof the fact that the Laplacian \( \Delta \) commutes with all rotations \( R_\theta \).)
Q8. Let $H$ be a real Hilbert space, let $K$ be a closed non-empty subset of $H$, and let $v$ be a point in $H$. Show that there exists a unique $w \in K$ which minimizes the distance to $v$ in the sense that $\|v - w\| < \|v - w'\|$ for all $w' \in K \setminus \{w\}$. (Hint: you may find the parallelogram law $\frac{\|a\|^2 + \|b\|^2}{2} = \|\frac{a+b}{2}\|^2 + \|\frac{a-b}{2}\|^2$ to be useful.)

Q9. Show using the residue theorem that

$$\int_0^\infty \frac{\log^2 x}{1 + x^2} = \frac{\pi^3}{8}.$$ 

Q10. Let the power series $f(z) = \sum_{n=0}^\infty a_n z^n$ have radius of convergence $r > 0$. For each $\rho$ with $0 < \rho < r$ let $M_f(\rho) := \sup\{|f(z)|; |z| = \rho\}$. Show that the following holds for each such $\rho$:

$$\sum_{n=0}^\infty |a_n|^2 \rho^{2n} \leq M_f(\rho)^2.$$ 

Q11. Let $\mathbb{D} := \{z; |z| < 1\}$ be the unit disc. Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map having 2 unequal fixed points $a, b \in \mathbb{D}$. Show that $f(z) = z$ for all $z \in D$. (Hint: use Schwartz's lemma.)

Q12. Consider the annulus $A := \{z \in \mathbb{C} : r < |z| < R\}$, where $0 < r < R$. Show that the function $f(z) := 1/z$ cannot be uniformly approximated in $A$ by complex polynomials.
Q13. Let $\Omega \subset \mathbb{C}$ be an open set containing the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$, and let $f_n : \Omega \to \mathbb{C}$ be a sequence of holomorphic functions on $\Omega$ which converge uniformly on compact subsets of $\Omega$ to a limit $f : \Omega \to \mathbb{C}$. Suppose that $|f(z)| \neq 0$ whenever $|z| = 1$. Show that there is a positive integer $N$ such that for $n \geq N$, the functions $f_n$ and $f$ have the same number of zeros in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. 