Analysis Qualifying Exam

Friday, September 19, 2014, 2:00 p.m.–6:00 p.m.

Students should solve four real analysis problems (numbered 1–6) and four complex analysis problems (numbered 7–12).

Problem 1. Show that

$$A := \{ f \in L^3(\mathbb{R}) : \int_{\mathbb{R}} |f(x)|^3 \, dx < \infty \}$$

is a Borel subset of $L^3(\mathbb{R})$.

Problem 2. Construct an $f \in L^1(\mathbb{R})$ so that $f(x + y)$ does not converge almost everywhere to $f(x)$ as $y \to 0$. Prove that your $f$ has this property.

Problem 3. Let $(f_n)$ be a bounded sequence in $L^2(\mathbb{R})$ and suppose that $f_n \to 0$ Lebesgue almost everywhere. Show that $f_n \rightharpoonup 0$ in the weak topology on $L^2(\mathbb{R})$.

Problem 4. Given $f \in L^2([0, \pi])$, we say that $f \in \mathcal{G}$ if $f$ admits a representation of the form

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(nx) \quad \text{with} \quad \sum_{n=0}^{\infty} (1 + n^2)|c_n|^2 < \infty.$$

Show that if $f \in \mathcal{G}$ and $g \in \mathcal{G}$ then $fg \in \mathcal{G}$.

Problem 5. Let $\phi : [0, 1] \to [0, 1]$ be continuous and let $d\mu$ be a Borel probability measure on $[0, 1]$. Suppose $\mu(\phi^{-1}(E)) = 0$ for every Borel set $E \subseteq [0, 1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $w : [0, 1] \to [0, \infty)$ so that

$$\int f \circ \phi(x) \, d\mu(x) = \int f(y)w(y) \, d\mu(y) \quad \text{for all continuous} \quad f : [0, 1] \to \mathbb{R}.$$

Problem 6. Let $X$ be a Banach space and $X^*$ its dual space. Suppose $X^*$ is separable (i.e. has a countable dense set); show that $X$ is separable. (You should assume the Axiom of Choice.)

Problem 7. Find an explicit conformal mapping from the upper half-plane slit along the vertical segment,

$$\{z \in \mathbb{C} : \text{Im } z > 0\} \setminus \{0, 0 + ih\}, \quad h > 0,$$

to the unit disc $\{z \in \mathbb{C} : |z| < 1\}$.

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that

$$|f'(z)| \leq C e^{\alpha |z|}, \quad z \in \mathbb{C},$$

for some constants $C$ and $\alpha$ if and only if we have

$$|f^{(n)}(0)| \leq M^{n+1}, \quad n = 0, 1, \ldots,$$
for some constant $M$.

**Problem 9.** Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose $(f_n)$ is a sequence of injective holomorphic functions defined on $\Omega$, such that $f_n \to f$ locally uniformly in $\Omega$. Show that if $f$ is not constant, then $f$ is also injective in $\Omega$.

**Problem 10.** Let us introduce a vector space $B$ defined as follows,

$$B = \left\{ u : \mathbb{C} \to \mathbb{C}, \quad u \text{ is holomorphic and } \iint_{\mathbb{C}} |u(x + iy)|^2 e^{-(x^2+y^2)} \, dx \, dy < \infty \right\}.$$

Show that $B$ becomes a complete space when equipped with the norm

$$\| u \|^2 = \iint_{\mathbb{C}} |u(x + iy)|^2 e^{-(x^2+y^2)} \, dx \, dy.$$

**Problem 11.** Let $\Omega \subset \mathbb{C}$ be open, bounded, and simply connected. Let $u$ be harmonic in $\Omega$ and assume that $u \geq 0$. Show the following: for each compact set $K \subset \Omega$, there exists a constant $C_K > 0$ such that

$$\sup_{x \in K} u(x) \leq C_K \inf_{x \in K} u(x). \tag{1}$$

**Problem 12.** Let $\Omega = \{ z \in \mathbb{C} : |z| > 1 \}$. Suppose $u : \overline{\Omega} \to \mathbb{R}$ is bounded and continuous on $\overline{\Omega}$ and that it is subharmonic on $\Omega$. Prove the following: If $u(z) \leq 0$ for all $|z| = 1$ then $u(z) \leq 0$ for all $z \in \Omega$. 