**Instructions:** Do all of the following:

1: Prove that $\mathbb{R}$ is uncountable. If you like to use the Baire category theorem, you have to prove it.

2: Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely often differentiable function. Assume that for each element $x \in [0, 1]$ there is a positive integer $m$, such that the $m$-th derivative of $f$ at $x$ is not zero.

Prove that there exists an integer $M$ such that the following stronger statement holds: For each element $x \in [0, 1]$ there is a positive integer $m$ with $m \leq M$ such that the $m$-th derivative of $f$ at $x$ is not zero.

3: Prove that the sequence $a_1, a_2, \ldots$ with

$$a_n = \left(1 + \frac{1}{n}\right)^n$$

converges as $n \to \infty$.

4: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. State the definition of the Riemann integral

$$\int_0^1 f(x) \, dx$$

and prove that it exists.

5: Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function such that all partial derivatives of order 3 exist and are continuous. Write down (explicitly in terms of partial derivatives of $f$) a quadratic polynomial $P(x, y)$ in $x$ and $y$ such that

$$|f(x, y) - P(x, y)| \leq C(x^2 + y^2)^{3/2}$$

for all $(x, y)$ in some small neighborhood of $(0, 0)$, where $C$ is a number that may depend on $f$ but not on $x$ and $y$. Then prove the above estimate.

6: Let $U = \{(x, y) : x^2 + y^2 < 1\}$ be the standard unit ball in $\mathbb{R}^2$ and let $\partial U$ denote its boundary.
2

Suppose \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) is continuously differentiable and that the Jacobian determinant of \( F \) is everywhere non-zero. Suppose also that \( F(x, y) \in U \) for some \((x, y) \in U\) and \( F(x, y) \not\in U \cup \partial U\) for all \((x, y) \in \partial U\). Prove that \( U \subset F(U) \).

7:

Prove that the space of continuous functions on the closed interval \([0, 1]\) with the metric
\[
\text{dist}(f, g) := \sup_{x \in [0,1]} |f(x) - g(x)| = \|f - g\|_{\infty}
\]
is complete. You do not need to show that this is a metric space.

8:

Prove the following three statements. You certainly may choose an order of these statements and then use the earlier statements to prove the later statements.

a) If \( T : V \to W \) is a linear transformation between two finite dimensional real vector spaces \( V, W \), then
\[
\dim(im(T)) = \dim(V) - \dim(\ker(T))
\]
b) If \( T : V \to V \) is a linear transformation on a finite dimensional real inner product space and \( T^* \) denotes its adjoint, then \( im(T^*) \) is the orthogonal complement of \( \ker(T) \) in \( V \).

c) Let \( A \) be a \( n \) by \( n \) real matrix, then the maximal number of linearly independent rows (row rank) in the matrix equals the maximal number of linearly independent columns (column rank).

9:

Consider a 3 by 3 real symmetric matrix with determinant 6. Assume \((1, 2, 3)\) and \((0, 3, -2)\) are eigenvectors with eigenvalues 1 and 2. Give answers to a) and b) below and justify the answers.

a) Give an eigenvector of the form \((1, x, y)\) for some real \(x, y\) which is linearly independent of the two vectors above.

b) What is the eigenvalue of this eigenvector.

10:

a) Let \( t \in \mathbb{R} \) such that \( t \) is not an integer multiple of \( \pi \). For the matrix
\[
A = \begin{pmatrix}
cos(t) & \sin(t) \\
-sin(t) & \cos(t)
\end{pmatrix}
\]
prove that there does not exist a real valued matrix \( B \) such that \( BAB^{-1} \) is a diagonal matrix.

b) Do the same for the matrix
\[
A = \begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}
\]
where \( \lambda \in \mathbb{R} \setminus \{0\} \).