1. (a) Suppose \( f : (0, 1) \to \mathbb{R} \) is a continuous function. Define what it means for \( f \) to be uniformly continuous.

(b) Show that if \( f : (0, 1) \to \mathbb{R} \) is uniformly continuous, then there is a continuous function \( F : [0, 1] \to \mathbb{R} \) with \( F(x) = f(x) \) for all \( x \in (0, 1) \).

2. Prove: If \( a_1, a_2, a_3, \ldots \) is a sequence of real numbers with
\[
\sum_{j=1}^{+\infty} |a_j| < +\infty,
\]
then \( \lim_{N \to +\infty} \sum_{j=1}^{N} a_j \) exists.

3. Find a subset \( S \) of the real numbers \( \mathbb{R} \) such that both (i) and (ii) hold for \( S \):

(i) \( S \) is not the countable union of closed sets

(ii) \( S \) is not the countable intersection of open sets.

4. Consider the following equation for a function \( F(x, y) \) on \( \mathbb{R}^2 \):
\[
\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} \quad (*)
\]

(a) Show that if a function \( F \) has the form \( F(x, y) = f(x + y) + g(x - y) \) where \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \) are twice differentiable, then \( F \) satisfies the equation (*)

(b) Show that if \( F(x, y) = ax^2 + bxy + cy^2 \), \( a, b, c \in \mathbb{R} \), satisfies (*) then \( F(x, y) = f(x, y) + g(x - y) \) for some polynomials \( f \) and \( g \) in one variable.

5. Consider the function \( F(x, y) = ax^2 + 2bxy + cy^2 \) on the set \( A = \{(x, y) : s^2 + y^2 = 1\} \).

(a) Show that \( F \) has a maximum and minimum on \( A \).

(b) Use Lagrange multipliers to show that if the maximum of \( F \) on \( A \) occurs at a point \((x_0, y_0)\), then the vector \((x_0, y_0)\) is an eigenvector of the matrix \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \).
6. Formulate some reasonably general conditions on a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which guarantee that
\[
\frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right)
\]
and prove that your conditions do in fact guarantee that this equality holds.

7. Let $V$ be a finite dimensional real vector space. If $W \subset V$ be a subspace let $W^o := \{ f : V \rightarrow \mathbb{F} \text{ linear}, |f = 0 \text{ on } W \}$. Let $W_i \subset V$ be subspaces for $i = 1, 2$. Prove that
   \[
   W_1^o \cap W_2^o = (W_1 + W_2)^o.
   \]

8. Let $V$ be an $n$-dimensional complex vector space and $T : V \rightarrow V$ a linear operator. Suppose that the characteristic polynomial of $T$ has $n$ distinct roots. Show that there is a basis $B$ for $V$ such that the matrix representation of $T$ is the basis $B$ is diagonal. (Make sure that you prove that your choice of $B$ is in fact a basis.)

9. Let $A \in \mathbf{M}_3(\mathbb{R})$ satisfy $\det(A) = 1$ and $A^t A = I = AA^t$ where $I$ is the $3 \times 3$ identity matrix. Prove that the characteristic polynomial of $A$ has 1 as a root.

10. Let $V$ be a finite dimensional real inner product space and $T : V \rightarrow V$ a hermitian linear operator. Suppose the matrix representation of $T^2$ in the standard basis has trace zero. Prove that $T$ is the zero operator.