Basic Exam, Fall 2013

Instructions: Write your UCLA student number on each page of your solutions. Do not write your name. Work 10 of the 12 problems, at least 4 of the first 6 and at least 4 of the last 6, and indicate here which 10 problems you want to have graded: (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12). Each problem is worth 10 points, but different parts of a problem may have different values.

1. When \( \{a_n\} \) is a sequence of positive real numbers, \( a_n > 0 \), define \( P_n = \prod_{j=1}^{n} (1 + a_j) \). Prove that \( \lim_{n \to \infty} P_n \) exists and is a non-zero real number if and only if \( \sum a_n < \infty \).

2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a nondecreasing real-valued function, i.e. if \( x < y \) then \( f(x) \leq f(y) \).
   
   (a) Prove that \( \{x \in \mathbb{R} : f \text{ is not continuous at } x\} \) is countable.

   (b) Let \( S \subset \mathbb{R} \) be a countable set. Prove there exists nondecreasing \( f : \mathbb{R} \to \mathbb{R} \) such that for all \( x \in \mathbb{R} \), \( f \) is not continuous at \( x \) if and only if \( x \in S \).

3. Let \( \gamma : [0,1] \to \mathbb{R}^2 \) be a continuous one-to-one function. By definition the length of the range \( \gamma([0,1]) \) is

\[
L(\gamma) = \sup \left\{ \sum_{j=1}^{n-1} |\gamma(t_{j+1}) - \gamma(t_j)| : 0 \leq t_1 < t_2 < \ldots < t_n \leq 1, \ n < \infty \right\}
\]

where \( |(x,y)| = \sqrt{x^2 + y^2} \) when \( (x,y) \in \mathbb{R}^2 \).

   (a) Suppose \( f(t) \) is continuous and nondecreasing on \( [0,1] \), and let \( \gamma(t) = (t, f(t)) \) (so that the range of \( \gamma \) is the graph of \( f \)). Prove

\[
L(\gamma) \leq 1 + (f(1) - f(0)).
\]

   (b) Show there exists continuous nondecreasing \( f(t) \) on \( [0,1] \) such that \( f(0) = 0 \) and \( f(1) = 1 \) and such that \( L(\gamma) = 2 \) when \( \gamma(t) = (t, f(t)) \).

4. Let \( f(x,y) \) be a continuous real-valued function on the plane \( \mathbb{R}^2 \). Assume that for every square \( S = \{a < x < a + \frac{1}{n}, b < y < b + \frac{1}{n}\} \), where \( a \) and \( b \) are rational numbers and \( n = 1, 2, \ldots, \)

\[
\int \int_S f(x,y) \, dx \, dy = 0.
\]

Prove \( f(x,y) = 0 \) for all \( (x,y) \).
5. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \text{ for all } x, y \in \mathbb{R}^d, \ 0 \leq t \leq 1.$$ 

Assume that $f$ is continuously differentiable such that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0, \ x, y \in \mathbb{R}^d$$

where $\nabla f$ is the gradient of $f$ and $\cdot$ is the inner product on $\mathbb{R}^d$. Prove that $f$ is convex.

6. Let $X$ be a compact metric space, let $\{x_n\}$ be a sequence in $X$ and let $x \in X$. Assume that for every subsequence $\{y_n\}$ of $\{x_n\}$ there is a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\{z_n\}$ converges to $x$. Prove $\{x_n\}$ converges to $x$.

7. Let $z_1, z_2, ..., z_n$ be distinct complex numbers and for $1 \leq j \leq n$, let $m_j$ be a non-negative integer. Write $N + 1 = \sum_{j=0}^{n}(1 + m_j)$. Prove that given any array of $N + 1$ complex numbers

$$c_{j,k}, \quad 1 \leq j \leq n, \quad 0 \leq k \leq m_j,$$

there is a unique polynomial $P(z)$ of degree at most $N$ such that for all $j, k$,

$$P^{(k)}(z_j) = c_{j,k}$$

where $P^{(k)}$ denotes the $k-$th derivative, i.e. $(z^n)^{(2)} = n(n - 1)z^{n-2}, n \geq 2$.

8. An orthogonal projection on a finite dimensional inner product space $V$ is an endomorphism $P$ that satisfies $P^2 = P$ and $im(P) = ker(P) \perp$. Suppose $V = \mathbb{R}^3$ and $P$ is an orthogonal projection with diagonal matrix entries $p_{1,1} = 2/3, p_{2,2} = 1/2, p_{3,3} = 5/6$. Find all matrices that $P$ could be. (Hint: it’s a very small finite set!).

9. Let $A$ be an endomorphism of a vector space $V$ of dimension $d$ over a field $F$. Show, from first principles (i.e. do not use Jordan form or the Cayley-Hamilton theorem) that $A$ satisfies a polynomial $P(X) \in F[X]$ of degree at most $d$.

10. Let $A$ be an $n$ by $n$ Hermitian matrix and let, for $j \in [1, n], A_j$ be the submatrix consisting of the entries of $A$ in the first $j$ row and columns of $A$. Suppose that $det(A_j) \neq 0$ and $det(A_1) > 0$. Give and prove a rule in terms of the signs of the $det(A_j)$ to determine the signature of the Hermitian form defined by $A$. 

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11. Define a normal linear transformation $N$ on a finite dimensional complex inner product space $V$. Supposing that $N$ is normal, show that there exists an orthogonal basis of $V$ consisting of eigenvectors of $N$.

12. Suppose $A$ is an endomorphism of a complex vector space with characteristic polynomial $C_A(X) = X^4 - 6X^3 + 13X^2 - 12X + 4$. How many similarity (i.e. conjugacy) classes of elements can have this characteristic polynomial? Suppose also that the minimal polynomial $M_A(X)$ of $A$ is equal to $C_A(X)$. How many classes satisfy this additional condition? Prove your answers, quoting any general theorems you need.