Basic Exam Spring 2014

IMPORTANT. Write your university identification number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam**

Test Instructions: Work 10 problems, including at least 4 from problems 1 - 6 and at least 4 from problems 7 - 12. Clearly indicate which 10 problems you have attempted. Usually a passing score requires very good work (at least 8 out of 10 points) on at least 3 problems from 1 - 6 and at least 3 problems from 7 - 12.

Problem Scores (NG=not graded)

1. Problem 1 _________
2. Problem 2 _________
3. Problem 3 _________
4. Problem 4 _________
5. Problem 5 _________
6. Problem 6 _________
7. Problem 7 _________
8. Problem 8 _________
9. Problem 9 _________
10. Problem 10 _________
11. Problem 11 _________
12. Problem 12 _________

Total _________
Notations:
Let $M_{m,n}(\mathbb{F})$ be the set of all $m$ by $n$ matrices with entries from the field $\mathbb{F}$. Let $\text{Hom}(U,V)$ be the set of all linear maps from the vector space $U$ to the the vector space $V$.

**Problem 1** (a) Find a real matrix $A$ whose minimal polynomial is equal to $t^4 + 1$.

(b) Show that the usual real linear map determined by $v \mapsto Av$ has no non-trivial invariant subspace.

**Problem 2** Suppose that $S, T \in \text{Hom}(V,V)$ where $V$ is a finite dimensional vector space over $\mathbb{R}$. Let $(\text{im} \ S)$ be the image of $S$ and $(\ker \ S)$ be the kernel of $S$. Show that

$$\dim(\text{im} \ S) + \dim(\text{im} \ T) \leq \dim(\text{im} \ S \circ T) + \dim V.$$ 

**Problem 3** Suppose that $A, B \in M_{n,n}(\mathbb{C})$ satisfy $AB - BA = A$. Show that $A$ is not invertible.

**Problem 4** Suppose that $A, B \in M_{n,n}(\mathbb{C})$. Show that the characteristic polynomials of $AB$ and $BA$ are equal. *Hint:* One approach is to first show that it holds when $B$ is invertible.

**Problem 5** Suppose that $V$ is a finite dimensional real inner product space with inner product $\langle \cdot, \cdot \rangle$, that $L \in \text{Hom}(V,V)$ and that $b \in V$ is fixed. Suppose that $u, v \in V$ both minimize $D(x) = ||L(x) - b||$. Show that $u - v \in \ker L$.

**Problem 6** Show that if $A \in M_{n,n}(\mathbb{C})$ is normal then $A^* = P(A)$ for some polynomial $P(x)$ with complex coefficients. Here $A^*$ is the conjugate transpose of $A$. 

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Problem 7 Find a doubly infinite sequence \( \{a_{n,m}, n, m \in \mathbb{Z}\} \) such that for all \( m \), \( \sum_n a_{n,m} = 0 \) and for all \( n \), \( \sum_m a_{n,m} = 0 \), with all these series converging absolutely, but such that \( \sum_n \sum_m |a_{n,m}| = \infty \).

Problem 8 (a) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
0, & t \leq 0 \\
e^{-\frac{1}{t}}, & t > 0.
\end{cases}
\]

Prove that \( f \) is infinitely differentiable.

(b) In Euclidean space \( \mathbb{R}^n \) for \( n \geq 1 \) find a function \( \varphi(x) \in C^\infty \) such that \( \varphi(x) \geq 0 \) for all \( x \), \( \varphi(x) = 0 \) if \( |x| > 1 \), and \( \int_{\mathbb{R}^n} \varphi(x)dx = 1 \).

Problem 9 Find a function that minimizes \( \int_0^1 |f'(x)|^2dx \) among all \( f \in C^1(\mathbb{R}) \) such that \( f(0) = 0, f(1) = 1 \). Is the minimizing \( C^1 \) function unique on \([0, 1]\)?

Problem 10 Let \( \mathcal{F} \) be a set of continuous real-valued functions on \([0, 1]\). Assume that

(i) \( \mathcal{F} \) is uniformly bounded, i.e. there is \( M < \infty \) such that \( |f(x)| \leq M \) for all \( f \in \mathcal{F} \) and all \( x \in [0, 1] \), and

(ii) \( \mathcal{F} \) is equicontinuous: for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that for all \( f \in \mathcal{F} \) and \( x, y \in [0, 1] \)

\[
|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.
\]

Prove that every sequence from \( \mathcal{F} \) has a subsequence that converges uniformly on \([0, 1]\).

Problem 11 Let \( \mathcal{F} \) be a set of continuous real-valued functions on \([0, 1]\). Assume that every sequence from \( \mathcal{F} \) has a subsequence that converges uniformly on \([0, 1]\). Prove both (i) and (ii) below hold:

(i) \( \mathcal{F} \) is uniformly bounded, i.e. there is \( M < \infty \) such that \( |f(x)| \leq M \) for all \( f \in \mathcal{F} \) and all \( x \in [0, 1] \), and

(ii) \( \mathcal{F} \) is equicontinuous: for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that for all \( f \in \mathcal{F} \) \( x, y \in [0, 1] \)

\[
|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.
\]
Problem 12 Assume \([0, 1] = \bigcup_{n=1}^{\infty} I_n\) where \(I_n = [a_n, b_n] \neq \emptyset\) and

\[I_n \cap I_m = \emptyset\]

whenever \(n \neq m\).

(a) Let \(E = \{a_n : n \geq 1\} \cup \{b_n : n \geq 1\}\) be the set of endpoints of the intervals above. Prove \(E\) is closed.

(b) Prove no such family of intervals \(\{I_n\}\) can exist.