Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. Do not write your name anywhere on the exam.

The final score will be the sum of **FOUR** analysis problems (Problems 1–6) and **FOUR** linear algebra problems (Problems 7–12). *On the front of your paper indicate which 8 problems you wish to have graded.* Please be reminded that to pass the exam you need to show mastery of both subjects.

Please staple your problems in the order they are listed in the exam.

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Problem 1. Let \( \{a_n\}_{n \geq 1} \) be a sequence of non-negative numbers such that
\[
\sum_{n \geq 1} a_n \text{ diverges.}
\]
Show that
\[
\sum_{n \geq 1} \frac{a_n}{2a_n + 1} \text{ diverges.}
\]

Problem 2. Let \( A \) be a connected subset of \( \mathbb{R}^n \) such that the complement of \( A \) is
the union of two separated sets \( B \) and \( C \), that is
\[
\mathbb{R}^n \setminus A = B \cup C \quad \text{with} \quad \overline{B} \cap C = B \cap \overline{C} = \emptyset.
\]
Show that \( A \cup B \) is a connected subset of \( \mathbb{R}^n \).

Problem 3. Let \( f : [0,1] \to \mathbb{R} \) and \( g : [0,1] \to [0,1] \) be two Riemann integrable
functions. Assume that
\[
|g(x) - g(y)| \geq \alpha |x - y| \quad \text{for any} \quad x, y \in [0,1]
\]
and some fixed \( \alpha \in (0,1) \). Show that \( f \circ g \) is Riemann integrable.

Problem 4. Let \( f : [0,1] \to \mathbb{R} \) be a continuous function on the closed interval
\([0,1]\) and differentiable on the open interval \((0,1)\). Assume that \( f(0) = 0 \) and \( f' \) is
a decreasing function on \((0,1)\). Show that
\[
g(x) = \frac{f(x)}{x}
\]
is a decreasing function on \((0,1)\).

Problem 5. Let \( B := \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \) and let \( g : \partial B \to \mathbb{R} \) be a 1-Lipschitz
function.
(a) Show that the function \( f : B \to \mathbb{R} \) given by
\[
f(x) := \inf_{y \in \partial B} \left[ g(y) + |x - y| \right]
\]
is 1-Lipschitz.
(b) Show that the set \( M(g) := \{ h : B \to \mathbb{R} \mid h \text{ is 1-Lipschitz and } h|_{\partial B} = g \} \) is compact in the space of continuous functions on \( B \) endowed with the supremum norm.

Problem 6. For \( x \in (0, \infty) \), let
\[
F(x) = \int_0^\infty \frac{1 - e^{-tx}}{t^2} \, dt.
\]
Show that \( F : (0, \infty) \to (0, \infty) \) is well-defined, bijective, of class \( C^1 \) (i.e. differentiable with continuous derivative), and that its inverse is of class \( C^1 \).
Problem 7. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a linear transformation with the property that 
\[
T(T(x)) = T(x)
\] for all \( x \in \mathbb{R}^n \).
Show that there exists \( 1 \leq m \leq n \) and a basis of \( \mathbb{R}^n \) such in this basis the entries of \( T \) satisfy
\[
T_{ij} = \begin{cases} 
1 & \text{if } i = j \text{ and } 1 \leq i \leq m, \\
0 & \text{otherwise}. 
\end{cases}
\]

Problem 8. Let \( X \) be an \( n \times n \) symmetric (real) matrix and \( z \in \mathbb{C} \) with \( \text{Im} \, z > 0 \).
Define
\[
G = (X - z)^{-1}.
\]
Show that
\[
\sum_{1 \leq j \leq n} |G_{ij}|^2 = \frac{\text{Im} \, G_{ii}}{\text{Im} \, z}.
\]

Problem 9. Let \( f, g : \mathbb{R}^n \to \mathbb{R} \) be linearly independent elements of the vector space (over \( \mathbb{R} \)) of linear mappings from \( \mathbb{R}^n \) to \( \mathbb{R} \). Show that for any \( v \in \mathbb{R}^n \), there exist \( v_1 \) and \( v_2 \) such that
\[
v = v_1 + v_2, \quad f(v) = f(v_1), \quad \text{and} \quad g(v) = g(v_2).
\]

Problem 10. Let \[
A := \begin{bmatrix} 
1 & 0 & 0 \\
1 & 2 & 1 \\
1 & 3 & 1 
\end{bmatrix}.
\]
Calculate \( \lim_{n \to \infty} A^n \).

Problem 11. Let \( V \) be the space of all \( 3 \times 3 \) real matrices that are skew-symmetric, i.e. \( A^t = -A \) (where \( A^t \) denotes the transpose of \( A \)). Prove that the expression
\[
\langle A, B \rangle = \frac{1}{2} \text{Tr}(AB^t)
\]
defines an inner product on \( V \). Exhibit an orthonormal basis of \( V \) with respect to this inner product; rigorously justify your answer.

Problem 12. Let \( V \) be a finite-dimensional vector space. Let \( T : V \to V \) be a linear transformation such that \( T(W) \subseteq W \) for every subspace \( W \) of \( V \) with \( \dim(W) = \dim(V) - 1 \). Prove that \( T \) is a scalar multiple of the identity operator.