1. Suppose $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are $C^\infty$ functions on $\mathbb{R}^3$ which vanish identically if $|x| \geq 5$, $|y| \geq 5$, or $|z| \geq 5$. Prove that the volume integral

$$\int_{-6}^{+6} \int_{-6}^{+6} \int_{-6}^{+6} d(P dy \wedge dz + Q dx \wedge dz + R dx \wedge dy) = 0.$$  

(Do this directly, not by quoting Stokes' Theorem: this is a special case of the proof of Stokes’ Theorem!)

2. Suppose that $V = P(x, y, z) \frac{\partial}{\partial x} + Q(x, y, z) \frac{\partial}{\partial y} + R(x, y, z) \frac{\partial}{\partial z}$ is a $C^\infty$ vector field on $\mathbb{R}^3$ with $V \neq \vec{0}$ at the origin. Find a necessary and sufficient condition for there to exist a $C^\infty$ function $\lambda(x, y, z)$ in some neighborhood of the origin such that $\lambda V$ is the gradient of a $C^\infty$ function on the neighborhood.

3. Let $T_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand rule rotation around the positive $z$-axis by $t$ degrees and $S_t : \mathbb{R}^3 \to \mathbb{R}^3$ be the right-hand-rule rotation around the positive $x$-axis by $t$ degrees.

   (a) Find the infinitesimal generators of the flows $T_t$ and $S_t$, i.e., the vector fields $X$ and $Y$, respectively, on $\mathbb{R}^3$ whose flows are $\{T_t\}$ and $\{S_t\}$.

   (b) Compute the commutator

   $$T_{-t} \circ S_{-t} \circ T_t \circ S_t.$$

   (c) Compare the result of (b) (lowest order non-identically zero term) with the Lie bracket $[X, Y]$.

4. Take as given that a $C^\infty$ 2-form $\omega$ on $S^2$ is of the form $d\theta$ for some $C^\infty$ 1-form $\theta$ if and only if $\int_{X^2} \omega = 0$. Use this to show that every $C^\infty$ 2-form $\Omega$ on $\mathbb{R}P^2$ has the form $d\Lambda$ for some $C^\infty$ 1-form $\Lambda$. (Do not just quote DeRham’s Theorem here.)

5. (a) Suppose $F : S^1 \to \mathbb{R}^3$ is a $C^\infty$ function such that $dF$ is nowhere zero (on $S^1$). Prove that there is a two-dimensional subspace $P$ of $\mathbb{R}^3$ such that $\pi_P \circ F : S^1 \to \mathbb{R}^3$ has nowhere vanishing differential, where $\pi_P$ = orthogonal projection on $P$.

   (b) Show by example (a picture with explanation is all right) that there is such an $F$ that is also 1 to 1 (injective) but is such that, for all $P$, $\pi_P \circ F$ fails to be injective.
(c) Show that if $F : S^1 \to \mathbb{R}^4$ is $C^\infty$ and injective then there is a three-dimensional subspace $H$ of $\mathbb{R}^4$ such that $\pi_H \circ F$ is injective, where $\pi_H$ = orthogonal projection on $H$.

6. (a) Suppose $F : S^n \to S^n$ is fixed-point free (i.e., for all $p \in S^n$, $p \neq F(p)$). Show that $F$ is homotopic to the antipodal map $p \to -p$, $p \in S^n$.

(b) Use part (a) to show that every vector field on (tangent to) $S^{2n}$, $n = 1, 2, 3 \ldots$, vanishes somewhere on $S^{2n}$ (i.e., has a zero).

7. (a) Discuss carefully how to obtain the long exact sequence in homology from a short exact sequence of chain complexes. (Include definitions of the maps in the long exact sequence.)

(b) If the short exact sequence is

$$0 \to C_1 \to C_2 \to C_3 \to 0,$$

prove exactness of the long exact sequence at $H_k(C_3)$ [in $\ldots H_k(C_2) \to H_k(C_3) \to H_{k-1}(C_1) \ldots$].

8. (a) Suppose $F : T^2 \to T^2$ (where $T^2 = S^1 \times S^1$) is a continuous function such that $F(p) = p$ for some $p \in T^2$ and

$$F_* : \pi_1(T^2, p) \to \pi_1(T^2, p)$$

is the identity map. Is $F$ necessarily homotopic to the identity map from $T^2$ to itself?

(b) Is a $C^\infty$ map $F : T^2 \to T^2$ of degree 1 necessarily homotopic to the identity map of $T^2$ to itself? Explain/prove your answer.

9. (a) Discuss the (a) representation of $CP^n$ as a cell complex.

(b) Use part (a) to find the homology of $CP^n$: prove carefully that your calculation is correct.

10. (a) Let $X =$ the space obtained by attaching two discs to $S^1$, the first disc being attached by $S^1 = \partial D_1 \to S^1$ being the 7 times around (counterclockwise) map, e.g., $z \to z^7$, $|z| = 1$, $z \in \mathbb{C}$ and the second being attached by $S^1 = \partial D_2 \to S^1$ being the 5 times around map $z \to z^5$. Find the homology of $X$.

(b) Can $X$ be made a $C^\infty$ manifold? Why or why not?