Instructions:

For a Ph.D. pass do 4 problems from each section to a total of 8 problems. For a M.A. pass do 2 from one and 3 from the other section to a total of 5 problems.

**Geometry**

1. Let $M$ be a closed (compact, without boundary) manifold. Show that any smooth function
   
   $$f : M \to \mathbb{R}$$

   has a critical point.

2. (a) Show that every closed 1-form on $S^n$, $n > 1$, is exact.

   (b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$, is exact.

3. Let $M^d$ be a $d$-dimensional manifold and $\omega_1, \ldots, \omega_p$ be pointwise linearly independent 1-forms. If $\theta_1, \ldots, \theta_p$ are 1-forms so that
   
   $$\sum_{i=1}^{p} \omega_i \wedge \theta_i = 0,$$

   then there exist smooth functions $f_{ij}$ so that
   
   $$\theta_i = \sum_{j=1}^{p} f_{ij} \omega_j, \quad i = 1, \ldots, p.$$

   **(Hint: try $p = 1$)**

4. Let $M$ be the set of all straight lines in $\mathbb{R}^2$ (not just those which pass through the origin). Show that $M$ is a smooth manifold and identify it with a well-known manifold.

   **(Hint: Lines not through the origin have a unique closest point to the origin and that point determines the line uniquely. What happens at the origin?)**

5. Let $f : M^n \to N^n$ be a smooth bijection so that $Df : T_p M \to T_{f(p)} N$ is injective for all $p$. Show that $f$ is a diffeomorphism.
6. (a) Show that if \( f : S^n \to S^n \) has no fixed points then \( \text{deg}(f) = (-1)^{n+1} \).

(b) Show that if \( X \) has \( S^{2n} \) as universal covering space then \( \pi_1(X) = \{1\} \) or \( \mathbb{Z}_2 \).

(c) Show that if \( X \) has \( S^{2n+1} \) as universal covering space then \( X \) is orientable.

7. (a) Outline the construction of the universal covering of a path connected locally simply connected space \( X \).

(b) Give an example of a path connected space which does not have a universal covering space.

8. Let \( X \) be a finite cell complex constructed inductively by gluing all \( p \)-cells onto cells of dimension \( < p \). Assume no \( p - 1 \) and \( p + 1 \) cells are used to construct \( X \). Show that

\[
H_p(X, \mathbb{Z}) \simeq \mathbb{Z}^{n_p}
\]

when \( n_p \) is the number of \( p \)-cells used in the construction.

9. Let \((M, \partial M)\) be a compact oriented \( n \)-manifold with connected boundary \( \partial M \). Show that there is no retract \( r : M \to \partial M \), i.e., a map \( r : M \to \partial M \) such that \( r(x) = x \) if \( x \in \partial M \).

(Hint: Prove that \( H_{n-1}(\partial M) \to H_{n-1}(M) \) is trivial.)

10. Let \( X = T^2 - \{p, q\} \), \( p \neq q \) be the twice punctured 2-dimensional torus.

(a) Compute the homology groups \( H_*(X, \mathbb{Z}) \).

(b) Compute the fundamental group of \( X \).