**Instruction:** All problems are worth ten points.

1. Let $M$ be a connected smooth manifold. Construct the orientation cover $M_0$.
   a) Show that $M_0$ is a smooth manifold.
   b) Show that $M_0$ is a 2:1 covering of $M$.
   c) Show that $M$ is orientable iff $M_0$ is the union of two disconnected components.

2. Let $\omega$ be a smooth nowhere vanishing 1-form on a smooth connected manifold $M$.
   a) Show that $\ker \omega$ is a smooth co-dimension 1 distribution on $M$.
   b) Show that $\ker \omega$ is integrable iff $d \omega$ vanishes on $\ker \omega$.
   c) Find a co-dimension 1 distribution on $\mathbb{R}^3$ that is not integrable.

3. Show that $S^1 \times S^n$ is parallelizable, i.e., one can find $(n + 1)$ vector fields that are everywhere linearly independent. ($S^k \subset \mathbb{R}^{k+1}$ is the unit sphere)

4. Let $\omega = \frac{-ydx + xdy}{(x^2 + y^2)\alpha}$ and consider $\int_{\gamma} \omega$, where $\gamma : S^1 \to \mathbb{R}^2 - \{0\}$.
   a) For which $\alpha$ is $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$, whenever $\gamma_1$ and $\gamma_2$ are smoothly homotopic, i.e., then exists $F : S^1 \times [0, 1] \to \mathbb{R}^2 - \{0\}$ such that $\gamma_0(t) = F(t, 0)$, $\gamma_1(t) = F(t, 1)$?
   b) What are the possible values for $\int_{\gamma} \omega$ when $\alpha$ is chosen as in part a)?

5. Show that a closed (compact without boundary) $n$-manifold cannot be immersed in $\mathbb{R}^n$. 

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6. Let \( \mathbb{C}^* \) be the set of all non-zero complex numbers with the induced topology from \( \mathbb{C} \). It is a topological group with respect to the usual multiplication. Let \( f \) be a continuous homomorphism from \( \mathbb{C}^* \) to itself.

   (i) Find all possible \( f|_{S^1} \), where \( S^1 = \{ z \mid |z| = 1, z \in \mathbb{C}^* \} \).

   (ii) Classify such \( f|_{S^1} \) up to homotopy.

7. Let \( X_1 = S^1 \vee_{x_1=x_2} S^2 \) be the space obtained from the disjoint union of the circle \( S^1 \) and the \( S^2 \) by identifying a point \( x_1 \in S^1 \) with a point \( x_2 \in S^2 \). Define \( X_2 = S^1 \vee_{x_1=x_2} S^1 \) similarly.

   (i) Find \( \pi_1(X_1) \) and \( \pi_1(X_2) \).

   (ii) Find their universal coverings.

8. Let \( f : S^2 \to T^2 \) be a continuous map from 2-sphere to 2-torus \( T^2 \).

   What is the induced map

   \[
   f_* : H_*(S^2) \to H_*(T^2)
   \]

   on the homology groups?

9. Let \( X \) be a topological space, and define \( S(X) \) to be the quotient space of \( X \times I \) by contracting \( X \times \{0\} \) to a point and \( X \times \{1\} \) to another point. Here \( I = [0,1] \).

   What is the relationship between \( H_*(S(x)) \) and \( H_*(x) \)?

10. Let \( K \) be a finite simplicial complex and \( K^n \) be the subcomplex consisting of all simplices in \( K \) of dimension less than or equal to \( n \). Denote the underlying topological spaces of \( K \) and \( K^n \) by \( |K| \) and \( |K^n| \).

   (i) What is the relative singular homology \( H_*(|K^n|,|K^{n-1}|) \)?

   (ii) Write down the long exact sequence for the triple \((|K^n|,|K^{n-1}|,|K^{n-2}|)\), i.e. , the long exact sequence relating the singular homology groups \( H_*(|K^n|,|K^{n-1}|) \), \( H_*(|K^{n-1}|,|K^{n-2}|) \) and \( H_*(|K^n|,|K^{n-2}|) \).

   (iii) Use (i) and (ii) to show that singular homology of \( |K| \) is same as the simplicial homology of \( |K| \). (Hint: identify the connecting boundary map in (ii)).