Manifold Problems

1. Let $M^2$ be a smooth 2-manifold and $f : M^2 \to \mathbb{R}$ be a smooth surjective map without critical points. Assume that for any finite closed interval $[a, b] \subseteq \mathbb{R}$, $f^{-1}([a, b])$ is compact. What is $M^2$?

2. Show that $T^2 \times S^2$ is parallelizable, i.e., there are 4 vector fields that are everywhere linearly independent.

3. Let $V = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z}$ be a nowhere zero $C^\infty$ vector field on $\mathbb{R}^3$. Show that the following three statements are equivalent.
   
   a) The orthogonal-to-$V$ plane field is integrable on some neighbourhood of $0 \in \mathbb{R}^3$.

   b) There exists a nowhere-zero $C^\infty$ function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\text{curl}(fV) \equiv 0$ on some neighbourhood of $0 \in \mathbb{R}^3$.

   c) $V \cdot \text{curl}(V) \equiv 0$ on some neighbourhood of $0 \in \mathbb{R}^3$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and $x \in \mathbb{R}^n$ be a critical point of $f$. The Hessian $H(t)_x$ at $x$ be a bilinear form: $T_x \mathbb{R}^n \times T_x \mathbb{R}^n \to \mathbb{R}$ defined as follows. For any two vectors $V_1$ and $V_2$ in $T_x \mathbb{R}^n$, extend $V_2$ to a vector field $\tilde{V}_2$ near $x$, and define $H(f)_x(V_1, V_2) = : D_{v_1}(D_{\tilde{v}_2} f)$.
Show that:

   (1) $H(f)_x(V_1, V_2) = H(f)_x(V_2, V_1)$.

   (2) $H(f)_x(V_1, V_2)$ is independent of the choice of the extension $\tilde{V}_2$.

5. (1) State Stokes' Theorem in its most general form.

   (2) Use the Stokes' Theorem to prove that for any vector field $X$ defined on $\mathbb{R}^n$,
   $\int_{\Omega} (\text{div} \, X) dx^1 \cdots dx^n = \pm \int_{\partial \Omega} (X \cdot N) ds$ where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$ and a unit normal field $N$ on $\partial \Omega$. Here $ds$ is the “area” form.
Topology Problems

1. Sketch the proof of:

**Theorem.** If $D$ is a subspace of $S^n$ homeomorphic to $I^k$ for some $k \geq 0$ then the reduced homology groups $\tilde{H}_i(S^n - D, \mathbb{Z})$ are trivial for all $i$.

(Hint: Induction on $k$.) (This is a special case of Alexander duality. No credit for saying "Applying Alexander duality ...".)

2. Show that $\mathbb{RP}^3$ is not homotopy equivalent to $\mathbb{RP}^2 \vee S^3$. (You could use cup products, degree, or covering spaces.)

3. Suppose $F : X \times I \to Y$ is a homotopy between $f : X \to Y$ and $g : X \to Y$. (6 pts) a) Indicate how to construct prism operators $P : C_n(X) \to C_{n+1}(Y)$ satisfying $g_* - f_* = \partial P + P \partial$ where $f_* : C_n(X) \to C_n(Y)$, $g_* : C_n(X) \to C_n(Y)$ are the chain maps.

(4 pts) b) Show that the induced homomorphisms $H_n(f)$, $H_n(g)$ are equal.

4. Give examples of a) two nonhomeomorphic connected regular 3-sheeted covering spaces of the bouquet of two circles and b) an irregular connected 3-sheeted cover of the bouquet of two circles.

5. (5 pts) a) Find the Euler characteristic of $X^2_4$, the 2-skeleton of the 4-simplex.

(5 pts) b) Give a reason why $H_2(X^2_4)$ is free abelian and find its rank.