Qualifying Exam
GEOMETRY-TOPOLOGY
March 2009

Instructions: Do any ten of the following twelve problems. Please do not turn in work on more than ten problems and label each problem carefully by its number. Start each problem on a new page.

1. (a) Show that a closed 1-form $\theta$ on $S^1 \times (-1,1)$ is $dF$ for some function $F: S^1 \times (-1,1) \to \mathbb{R}$ if and only if $\int_{S^1} i^* \theta = 0$ where $i: S^1 \to S^1 \times (-1,1)$ is defined by $i(p) = (p,0)$ for $p \in S^1$. (b) Show that a 2-form $\omega$ on $S^2$ is $d\theta$ for some 1-form $\theta$ on $S^1$ if and only if $\int_{S^2} \omega = 0$.

2. Suppose that $M, N$ are connected $C^\infty$ manifolds of the same dimension $n \geq 1$ and $F: M \to N$ is a $C^\infty$ map such that $dF: T_p M \to T_{F(p)} N$ is surjective for each $p \in M$. (a) Prove that if $M$ is compact, then $F$ is onto and $F$ is a covering map. (b) Find an example of such an everywhere nonsingular equidimensional map where $N$ is compact, $F$ is onto, $F^{-1}(p)$ is finite for each $p \in N$, but $F$ is not a covering map. [A clearly explained pictorial version of $F$ will be acceptable; you do not need to have a "formula" for $F$.]

3. (a) Suppose that $M$ is a $C^\infty$ connected manifold. Prove that, given an open subset $U$ of $M$ and a finite set of points $p_1, p_2, \ldots, p_k$ in $U$, there is a diffeomorphism $F: M \to M$ such that $f(\{p_1, p_2, \ldots, p_k\}) \subset U$. [Suggestion: Construct $F$ one point at a time.] (b) Use part (a) to show that if $M$ is compact and the Euler characteristic $\chi(M) = 0$, then there is a vector field on $M$ which vanishes nowhere. You may assume that if a vector field has isolated zeros, then the sum of the indices at the zero points equals $\chi(M)$.

4. A smooth vector field $V$ on $\mathbb{R}^3$ is said to be "gradient-like" if, for each $p \in \mathbb{R}^3$, there is a neighborhood $U_p$ of $p$ and a function $\lambda_p: U_p \to \mathbb{R} - \{0\}$ such that $\lambda_p V$ on $U_p$ is the gradient of some $C^\infty$ function on $U_p$. Suppose $V$ is nowhere zero on $\mathbb{R}^3$. Then show that $V$ is gradient-like if and only if $\text{curl} V$ is perpendicular to $V$ at each point of $\mathbb{R}^3$.

5. Suppose that $M$ is a compact $C^\infty$ manifold of dimension $n$. (a) Show that there is a positive integer $k$ such that there is an immersion $F: M \to \mathbb{R}^k$. (b) Show that if $k > 2n$, there is a $(k-1)$-dimensional subspace $H$ of $\mathbb{R}^k$ such that $P \circ F$ is an immersion, where $P: \mathbb{R}^k \to H$ is orthogonal projection.

6. Let $GL^+(n, \mathbb{R})$ be the set of $n \times n$ matrices with determinant $> 0$. Note that $GL^+(n, \mathbb{R})$ can be considered to be a subset of $\mathbb{R}^{n^2}$ and this subset is open. (a) Prove that $SL^+(n, \mathbb{R}) = \{ A \in GL^+(n, \mathbb{R}): \det A = 1 \}$ is a submanifold. (b) Identify the tangent space of $SL^+(n, \mathbb{R})$ at the identity matrix $I_n$. (c) Prove that, for every $n \times n$ matrix $B$, the series $I_n + B + \frac{1}{2} B^2 + \frac{1}{3!} B^3 + \cdots + \frac{1}{n!} B^n \cdots$ converges to some $n \times n$ matrix. Notation: this sum $= e^B$. (d) Prove that if $e^{tB} \in SL^+(n, \mathbb{R})$ for all $t \in \mathbb{R}$,
then \( \text{trace } B = 0 \). (e) Prove that if \( \text{trace } B = 0 \), then \( e^B \in SL^+(n, \mathbb{R}) \). [Suggestion: Use one-parameter subgroups or note that it suffices to treat complex-diagonale \( B \) since such are dense.]

7. (a) Define complex projective space \( \mathbb{CP}^n \). (b) Calculate the homology of \( \mathbb{CP}^n \). Any systematic method such as Mayer-Vietoris or cellular homology is acceptable.

8. Let \( p: E \to B \) be a covering space and \( f: X \to B \) a map. Define \( E^* = \{(x, e) \in X \times B : f(x) = p(e)\} \). Prove that \( q: E^* \to X \) defined by \( q(x, e) = x \) is a covering space.

9. (a) Explain carefully and concretely what it means for two (smooth) maps of \( S^1 \) into \( \mathbb{R}^2 \) to be transversal. (b) Do the same for maps of \( S^1 \) into \( \mathbb{R}^3 \). (c) Explain what it means for transversal maps to be "generic" and prove that they are indeed generic in the cases of 9(a) and 9(b).

10. Let \( M \) be the 3-manifold with boundary obtained as the union of the two-holed torus in 3-space and the bounded component of its complement. Let \( X \) be the space obtained from \( M \) by deleting \( k \) points from the interior of \( M \). (a) Calculate the fundamental group of \( X \). (b) Calculate the homology of \( X \).

11. Let \( P \) be a finite polyhedron. (a) Define the Euler characteristic \( \chi(\mathcal{P}) \) of \( \mathcal{P} \). (b) Prove that if \( P_1, P_2 \) are subpolyhedra of \( \mathcal{P} \) such that \( P_1 \cap P_2 \) is a point and \( P_1 \cup P_2 = \mathcal{P} \), then \( \chi(\mathcal{P}) = \chi(P_1) + \chi(P_2) - 1 \). (c) Suppose that \( p: E \to \mathcal{P} \) is an \( n \)-sheeted covering space of \( \mathcal{P} \), that is \( p^{-1}(x) \) is \( n \) points for each \( x \in \mathcal{P} \). Prove that \( \chi(E) = n\chi(\mathcal{P}) \).

12. Let \( f: T \to T = S^1 \times S^1 \) be a map of the torus inducing \( f_\pi: \pi_1(T) \to \pi_1(T) = \mathbb{Z} \oplus \mathbb{Z} \) and let \( F \) be a matrix representing \( f_\pi \). Prove that the determinant of \( F \) equals the degree of the map of the map \( f \).