Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $f : M \to N$ be a nonsingular smooth map between connected manifolds of the same dimension. Answer the following questions with a proof or counter-example.
   (a) Is $f$ necessarily injective or surjective?
   (b) Is $f$ necessarily a covering map when $N$ is compact?
   (c) Is $f$ necessarily an open map?
   (d) Is $f$ necessarily a closed map?

2. Let $M$ be a connected compact manifold with non-empty boundary $\partial M$. Show that $M$ does not retract onto $\partial M$.

3. Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions $m$ and $n$, respectively, such that $m + n = p$. Suppose that $M \cap N = \emptyset$. Consider the linking map
   $$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{\|x - y\|}.$$ 
   The degree of $\lambda$ is called the linking number $l(M, N)$.
   (a) Show that $l(M, N) = (-1)^{(m+1)(n+1)}l(N, M)$.
   (b) Show that if $M$ is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from $N$, then $l(M, N) = 0$.

4. Let $\omega$ be a 1-form on a connected manifold $M$. Show that $\omega$ is exact, i.e., $\omega = df$ for some function $f$, if and only if for all piecewise smooth closed curves $c : S^1 \to M$ it follows that $\int_c \omega = 0$.

5. Let $\omega$ be a smooth, nowhere vanishing 1-form on a three-dimensional smooth manifold $M^3$.
   (a) Show that $\text{ker } \omega$ is an integrable distribution on $M$ if and only if $\omega \wedge d\omega = 0$.
   (b) Give an example of a codimension one distribution on $\mathbb{R}^3$ that is not integrable.
6. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth function.

(a) Define the gradient \( \nabla f \) as a vector field dual to the differential \( df \).

(b) Define the Hessian \( \text{Hess} f (X, Y) \) as a symmetric \((0, 2)\)- tensor.

(c) If the usual Euclidean inner product between tangent vectors in \( T_p \mathbb{R}^n \) is denoted \( g (X, Y) = X \cdot Y \) show that
\[
\text{Hess} f (X, Y) = \frac{1}{2} (\mathcal{L}_{\nabla f} g) (X, Y)
\]
Here \( \mathcal{L}_Z g \) is the Lie derivative of \( g \) in the direction of \( Z \).

7. Let \( M = T^2 - D^2 \) be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of \( M \).

8. Let \( n > 0 \) be an integer and let \( A \) be an abelian group with a finite presentation by generators and relations. Show that there exists a topological space \( X \) with \( H_n(X) \cong A \).

9. Let \( H \subset S^3 \) be the Hopf link, shown in the figure

\[
\begin{tikzpicture}
  \draw (0,0) circle (1);
  \draw (1,0) circle (1);
  \draw (0.5,0) -- (0.5,1);
\end{tikzpicture}
\]

Compute the fundamental group and the homology groups of the complement \( S^3 - H \).

10. Let \( \mathbb{H} = \mathbb{R} \oplus \mathbb{R} i \oplus \mathbb{R} j \oplus \mathbb{R} k \) be the group of quaternions, with relations \( i^2 = j^2 = -1, \ ij = -ji = k \). The multiplicative group \( \mathbb{H}^* = \mathbb{H} - \{0\} \) acts on \( \mathbb{H}^n - \{0\} \) by left multiplication. The quotient \( \mathbb{H}P^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^* \) is called the quaternionic projective space. Calculate its homology groups.