Answer all 10 questions. Each problem is worth 10 points. Justify your answers carefully.

1. Let $M(n, m, k) \subset M(n, m)$ denote the space of $n \times m$-matrices of rank $k$. Show that $M(n, m, k)$ is a smooth manifold of dimension $nm - (n - k)(m - k)$.

2. Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that $N$ can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f: M \to \mathbb{R}$, if and only if there is a vector field $X$ on $M$ that is transverse to $N$.

3. Consider two collections of 1-forms $\omega_1, \ldots, \omega_k$ and $\phi_1, \ldots, \phi_k$ on an $n$-dimensional manifold $M$. Assume that

$$\omega_1 \wedge \cdots \wedge \omega_k = \phi_1 \wedge \cdots \wedge \phi_k$$

never vanishes on $M$. Show that there are smooth functions $f_{ij}: M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{k} f_{ij} \phi_j, \quad i = 1, \ldots, k.$$

4. Consider a smooth map $F: \mathbb{RP}^n \to \mathbb{RP}^n$.

(a) When $n$ is even show that $F$ has a fixed point.

(b) When $n$ is odd give an example where $F$ does not have a fixed point.

5. Assume we have a codimension 1 distribution $\Delta \subset TM$.

(a) Show if the quotient bundle $TM/\Delta$ is trivial (or equivalently that there is a vector field on $M$ that never lies in $\Delta$), then there is a 1-form $\omega$ on $M$ such that $\Delta = \ker \omega$ everywhere on $M$.

(b) Give an example where $TM/\Delta$ is not trivial.

(c) With $\omega_1$ and $\omega_2$ as in (a) show that $\omega_1 \wedge d\omega_1 = f^2 \omega_2 \wedge d\omega_2$ for a smooth function $f$ that never vanishes.

(d) If $\omega$ is as in (a) and $\omega \wedge d\omega \neq 0$, show that $\Delta$ is not integrable.

6. Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

be a 2-form defined on $\mathbb{R}^3 - \{0\}$. If $i: S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbb{R}^3$ is the inclusion, then compute $\int_{S^2} i^* \omega$. Also compute $\int_{S^2} j^* \omega$, where $j: S^2 \to \mathbb{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$. 

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**Question 1:**

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**Question 2:**

Assume that $N \subset M$ is a codimension 1 properly embedded submanifold. Show that $N$ can be written as $f^{-1}(0)$, where 0 is a regular value of a smooth function $f: M \to \mathbb{R}$, if and only if there is a vector field $X$ on $M$ that is transverse to $N$.

**Question 3:**

Consider two collections of 1-forms $\omega_1, \ldots, \omega_k$ and $\phi_1, \ldots, \phi_k$ on an $n$-dimensional manifold $M$. Assume that

$$\omega_1 \wedge \cdots \wedge \omega_k = \phi_1 \wedge \cdots \wedge \phi_k$$

never vanishes on $M$. Show that there are smooth functions $f_{ij}: M \to \mathbb{R}$ such that

$$\omega_i = \sum_{j=1}^{k} f_{ij} \phi_j, \quad i = 1, \ldots, k.$$

**Question 4:**

Consider a smooth map $F: \mathbb{RP}^n \to \mathbb{RP}^n$.

(a) When $n$ is even show that $F$ has a fixed point.

(b) When $n$ is odd give an example where $F$ does not have a fixed point.

**Question 5:**

Assume we have a codimension 1 distribution $\Delta \subset TM$.

(a) Show if the quotient bundle $TM/\Delta$ is trivial (or equivalently that there is a vector field on $M$ that never lies in $\Delta$), then there is a 1-form $\omega$ on $M$ such that $\Delta = \ker \omega$ everywhere on $M$.

(b) Give an example where $TM/\Delta$ is not trivial.

(c) With $\omega_1$ and $\omega_2$ as in (a) show that $\omega_1 \wedge d\omega_1 = f^2 \omega_2 \wedge d\omega_2$ for a smooth function $f$ that never vanishes.

(d) If $\omega$ is as in (a) and $\omega \wedge d\omega \neq 0$, show that $\Delta$ is not integrable.

**Question 6:**

Let

$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

be a 2-form defined on $\mathbb{R}^3 - \{0\}$. If $i: S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbb{R}^3$ is the inclusion, then compute $\int_{S^2} i^* \omega$. Also compute $\int_{S^2} j^* \omega$, where $j: S^2 \to \mathbb{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$. 

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7. Define the de Rham cohomology groups $H^i_{dR}(M)$ of a manifold $M$ and compute $H^i_{dR}(S^1)$, $S^1 = \mathbb{R}/\mathbb{Z}$, $i = 0, 1, \ldots$, directly from the definition.

8. Let $X$ be a CW complex consisting one vertex $p$, 2 edges $a$ and $b$, and two 2-cells $f_1$ and $f_2$, where the boundaries of $a$ and $b$ map to $p$, the boundary of $f_1$ is mapped to the loop $ab^2$ (that is first $a$ and then $b$ twice), and the boundary of $f_2$ is mapped to the loop $ba^2$.

(a) Compute the fundamental group $\pi_1(X)$ of $X$. Is it a finite group?
(b) Compute the homology groups $H_i(X)$, $i = 0, 1, \ldots$, of $X$.

9. Let $X, Y$ be topological spaces and let $f, g : X \to Y$ be two continuous maps. Consider the space $Z$ obtained from the disjoint union $(X \times [0, 1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:

$$\cdots \to H_i(X) \xrightarrow{a} H_i(Y) \xrightarrow{b} H_i(Z) \xrightarrow{c} H_{i-1}(X) \to \cdots$$

Also describe the maps $a$, $b$, $c$.

10. Let $n \geq 0$ be an integer. Let $M$ be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim H^{2n+1}(M; \mathbb{R})$ is even.