Attempt all ten problems. Each problem is worth 10 points. Justify your answers carefully.

1. Suppose that $M$ and $N$ are connected smooth manifolds of the same dimension and $f : M \to N$ is a smooth submersion.
   (a) Prove that if $M$ is compact, then $f$ is onto and $f$ is a covering map.
   (b) Give an example of a smooth submersion $f : M \to N$ such that $M$ and $N$ have the same dimension, $N$ is compact, and $f$ is onto, but $f$ is not a covering map.

2. Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global flows on the sphere $S^2$. Show that there exist $\epsilon > 0$, a neighborhood $U$ of the North pole, a neighborhood $V$ of the South pole, and a global flow $\Phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in U$, and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon), q \in V$.

3. For $n \geq 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by
   $$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_{n+1} = z_{n+2} = \cdots = z_{2n} = 0\}.$$ 
   (a) Show that $X$ is a smooth submanifold.
   (b) Calculate the mod 2 intersection number of $X$ with itself.

4. Suppose $N$ is a smoothly embedded submanifold of a smooth manifold $M$. A vector field $X$ on $M$ is called tangent to $N$ if $X_p \in T_p N \subset T_p M$ for all $p \in M$.
   (a) Show that if $X$ and $Y$ are vector fields on $M$ both tangent to $N$, then $[X, Y]$ is also tangent to $N$.
   (b) Illustrate this principle by choosing two vector fields $X, Y$ tangent to $S^2 \subset \mathbb{R}^3$ (such that $[X, Y]$ is not identically zero), computing $[X, Y]$ and checking that it is tangent to $S^2$.

5. A symplectic form on an eight-dimensional manifold is defined to be a closed two-form $\omega$ such that $\omega \wedge \omega \wedge \omega \wedge \omega$ is a volume form (that is, everywhere nonvanishing). Determine which of the following manifolds admit symplectic forms: (a) $S^8$; (b) $S^2 \times S^6$; (c) $S^2 \times S^2 \times S^2 \times S^2$.

6. Let $U$ be a bounded open set in $\mathbb{R}^3$ with smooth boundary, and let $V$ be a smooth vector field on $\mathbb{R}^3$. The classical divergence theorem expresses the triple integral $\iiint_V \text{div} V \, d(\text{vol})$ as a surface integral over the boundary of $V$. State this theorem, and show how it can be obtained as a particular case of Stokes’ Theorem for differential forms.

7. Let $M$ and $N$ be smooth, connected, orientable $n$-manifolds for $n \geq 3$, and let $M \# N$ denote their connect sum.
   (a) Compute the fundamental group of $M \# N$ in terms of that of $M$ and of $N$ (you may assume that the basepoint is on the boundary sphere along which we glue $M$ and $N$).
(b) Compute the homology groups of \( M \# N \). (You may use without proof that \( H_n(-; \mathbb{Z}) \) of a connected orientable \( n \)-manifold is always isomorphic to \( \mathbb{Z} \)).

(c) For part (a), what changes if \( n = 2 \)? Use this to describe the fundamental groups of orientable surfaces.

8. Determine all of the possible degrees of maps \( S^2 \to S^1 \times S^1 \).

9. Point \( S^2 \) via the south pole, and consider the Cartesian product \( S^2 \times S^2 \).

(a) Describe a cell structure on \( S^2 \times S^2 \) that is compatible with the inclusion of

\[ S^2 \vee S^2 \hookrightarrow S^2 \times S^2 \]

as those pairs where one coordinate is the south pole.

(b) Let \( X \) be \( (S^2 \times S^2) \cup_{S^2} D^3 \), where we attach the 3-disk via the map

\[ S^2 \to S^2 \vee S^2 \]

which crushes a great circle connecting the north and south poles. Compute the homology groups of \( X \).

10. Let \( X \) be a semi-locally simply connected space and let \( \tilde{X} \to X \) be the universal cover.

(a) Show that any map \( \sigma: \Delta^n \to X \) lifts to a map \( \tilde{\sigma}: \Delta^n \to \tilde{X} \), where \( \Delta^n \) is the standard \( n \)-simplex.

(b) Show that if \( \tilde{\sigma}_1, \tilde{\sigma}_2: \Delta^n \to \tilde{X} \) are two lifts of \( \sigma \), then there is an element \( g \) of the fundamental group of \( X \) such that \( g \circ \tilde{\sigma}_1 = \tilde{\sigma}_2 \), where we view \( g \) as an automorphism of \( \tilde{X} \) via the deck transformations.