All problems have equal value. Solution of four problems will ensure an M.A. pass.

1. As usual, for each number \( e \in \mathbb{N} \),
   \[ W_e = \{ x \mid \phi_e(x) \downarrow \} \]
where \( \phi_0, \phi_1, \ldots \) is a standard enumeration of all recursive partial functions.

(1a) Prove that the set
   \[ A = \{ e \mid W_e \text{ is finite} \} \]
is in \( \Sigma_2 \setminus \Pi_2 \).

(1b) Classify in the arithmetical hierarchy the set
   \[ B = \{ e \mid W_e \text{ is finite and has an even number of elements} \} \]
i.e., find some \( n \) such that \( B \in \Sigma_n \setminus \Pi_n \) or \( B \in \Pi_n \setminus \Sigma_n \).

2. (2a) Prove that for every recursive partial function \( f(x, y) \), there is some recursive partial function \( g(x) \) such that
   (A) \[ g(x) \downarrow \iff (\exists y)[f(x, y) \downarrow] \]
   (B) \[ (\exists y)[f(x, y) \downarrow] \implies f(x, g(x)) \downarrow \]

(2b) Show that we cannot strengthen (2a) by replacing (B) by the stronger
     (B') \[ (\exists y)[f(x, y) \downarrow] \implies g(x) = \text{the least } y \text{ such that } f(x, y) \downarrow \]

3. For each sentence \( \theta \) in the language of Peano arithmetic \( \text{PA} \), let
   \[ ^\gamma \theta = \text{the (formal) numeral of the Gödel number of } \theta, \]
and let \( \text{Pr}(n) \) be a formula with one free variable which expresses the relation of provability in Peano arithmetic, so that (in particular), for each sentence \( \theta \),
   \[ (\mathbb{N}, 0, 1, +, \cdot) \models \text{Pr}(^\gamma \theta) \iff \text{PA} \vdash \theta. \]
Consider the following four sentences which can be constructed from an arbitrary sentence \( \theta \):
   (a) \( \theta \rightarrow \text{Pr}(^\gamma \theta) \)
   (b) \( \text{Pr}(^\gamma \theta) \rightarrow \theta \)
   (c) \( \text{Pr}(^\gamma \theta) \rightarrow \text{Pr}(^\gamma \text{Pr}(^\gamma \theta)) \)
   (d) \( \text{Pr}(^\gamma \text{Pr}(^\gamma \theta)) \rightarrow \text{Pr}(^\gamma \theta) \)
Determine which of these four sentences are provable in \( \text{PA} \) (for every choice of \( \theta \)), and justif\(y three of your answers\) by appealing, if necessary, to standard theorems which are proved in 220. (One of the answers is more difficult to justify than the others.)
4. Assume \( V = L \), let
\[ \lambda = \aleph_\omega, \]
and prove that \( L_\lambda \) has the \( \Sigma_1 \)-reflection property.

In detail, this means that if
\[ \theta(x, y) \equiv (\exists x_1)(\exists x_2) \cdots (\exists x_n) \phi(x, y) \]
where \( \phi(x, y) \) is a bounded formula in which all quantifiers occur in one of the forms
\[ (\exists y \in z) \text{ or } (\forall y \in z), \]
and if for some \( a \in L_\lambda \),
\[ L_\lambda \models (\forall x \in a)(\exists y \in b) \theta(x, y), \]
then there is some \( b \in L_\lambda \) such that
\[ L_\lambda \models (\forall x \in a)(\exists y \in b) \theta(x, y). \]

5. Suppose \( V = L \). True or false: if \( \alpha \) is an ordinal such that
\[ L_\alpha \models \text{ZFC}, \]
then \( \alpha \) is a strongly inaccessible cardinal. (You must prove your answer.)

6. A tree on a set \( A \) is a set \( T \subseteq A^{<\omega} \) of finite sequences from \( A \) which is closed under initial segments; an infinite branch of a tree \( T \) is any function \( \alpha : \mathbb{N} \to A \) such that for all \( n \), \( \langle \alpha(0) < \ldots < \alpha(n-1) \rangle \in T \); and \( T \) is finitely splitting if for each \( u \in T \) there are only finitely many (perhaps 0) one-point extensions of \( u \) in \( T \). Prove the following

König's Lemma. Every infinite, finitely splitting tree \( T \) has an infinite branch.

7. Recall that a model \( \mathcal{M} \) of a complete (first-order) theory \( T \) is atomic, if for every finite sequence \( \vec{a} \in M^n \) of length \( n \), there is a formula \( \phi(\vec{v}) \) with \( n \) free variables such that
\[ \mathcal{M} \models \phi(\vec{a}), \]
and for every \( \psi(\vec{v}) \),
either \[ \mathcal{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \psi(\vec{v})] \text{ or } \mathcal{M} \models (\forall \vec{v})[\phi(\vec{v}) \to \neg \psi(\vec{v})] \]

(7a). Does there exist a countable, complete theory with an atomic model of size \( \aleph_0 \) but no atomic model of size \( \aleph_1 \)?

(7b). Does there exist a countable, complete theory with an atomic model of size \( \aleph_1 \) but no atomic model of size \( \aleph_0 \)?

8. Let \( \mathcal{M} = (\mathbb{Z}, S) \) be the model with underlying set the integers and the successor function \( S(x) = x + 1 \) as the only non-logical constant. Is \( \text{Th}(\mathcal{M}) \) finitely axiomatizable?

9. Let \( T = \text{Th}(\mathbb{R}, <, \mathbb{Z}) \) be the theory of the real numbers, with the usual ordering and a distinguished predicate for the integers. Is \( T \aleph_0 \)-categorical? (You must prove your answer.)