Problem 1. Prove that there is no formula \( \varphi(v, u) \) so that \((\mathbb{N}; 0, 1, +) \models \varphi[x, y] \) iff \( x \) divides \( y \).

Problem 2. Let \( \mathcal{L} \) be the language of graphs; the only non-logical symbol of the language is a binary relation symbol, \( E \).

(2a) For which \( n < \omega \) is there a finitely axiomatizable theory \( T_n \) in \( \mathcal{L} \), so that the models of \( T_n \) are exactly the \( n \)-colorable graphs?

(2b) Construct an \( \omega \)-saturated model \( A = (A; E_A) \) of \( \mathcal{L} \), so that for every \( n < \omega \) there is a definable (from parameters) set \( D_n \subseteq A \), so that \((D_n; E_A \upharpoonright D_n) \) forms a graph which is \( n + 1 \)-colorable but not \( n \)-colorable.

Problem 3. Say that a formula \( \varphi(v_1, \ldots, v_l, u) \) in the language of set theory defines a set \( x \) from parameters \( a_1, \ldots, a_l, \) and \( x \) is definable from \( a_1, \ldots, a_l \), if \( \varphi(a_1, \ldots, a_l, x) \) holds (in \( V \)) and for every \( y \neq x, \varphi(a_1, \ldots, a_l, y) \) fails.

(3a) Suppose there is a model of \( \text{ZFC} \). Prove there is a model of \( \text{ZFC} \) in which every set is definable from ordinal parameters.

(3b) Prove that if \( x \) is definable from ordinal parameters, then it is definable from ordinal parameters by a \( \Sigma_2 \) formula.

Problem 4. Assume \( \text{AC} \). Assume that \( \aleph_1^{\aleph_0} = \aleph_2 \) and \( (\forall \alpha < \aleph_2) \alpha^{\aleph_0} = \aleph_1 \) (under \( \text{AC} \) these are consequences of the GCH).

(4a) Prove that there is a sequence \( \langle A_\xi \mid \xi < \omega_2 \rangle \) of bounded subsets of \( \aleph_2 \) so that for every \( A \subseteq \aleph_2 \), there is a club of \( \gamma < \aleph_2 \) so that \((\forall \beta < \gamma) A \cap \beta \in \{A_\xi \mid \xi < \gamma \} \).

(4b) Prove that there is a sequence \( \langle F_\alpha \mid \alpha < \omega_2, \text{cof}(\alpha) = \omega \rangle \) so that each \( F_\alpha \) is a family of subsets of \( \alpha, |F_\alpha| \leq \aleph_1 \), and for each \( A \subseteq \aleph_2 \), \( \{\alpha < \aleph_2 \mid A \cap \alpha \in F_\alpha \} \) is stationary.

Problem 5. Let \( \varphi_i \) be the \( i \)-th partial recursive function from \( \omega \) to \( \omega \). Let \( T = \{i : \varphi_i \text{ is total}\} \). Show that every \( \Pi_2 \) subset of \( \omega \) is many-one reducible to \( T \).

Problem 6. Let \( W_i \) be the \( i \)-th computably enumerable set. Say that \( i \) is a minimal index if for all \( j < i \), \( W_j \neq W_i \).

(6a) Show that \( \{i : i \text{ is a minimal index}\} \) contains no infinite computably enumerable subset.

(6b) Show that there are only finitely many \( i \) such that \( \text{PA} \) proves “\( i \) is a minimal index”.

Problem 7. Let \( \text{Prov}_{\text{PA}}(n) \) be the formula of arithmetic asserting that there is a proof of the formula with \( \text{G"{o}del} \) number \( n \) from \( \text{PA} \). Let \( \ulcorner \varphi \urcorner \) indicate the (numeral of the) \( \text{G"{o}del} \) number of the formula \( \varphi \). You may use that the provability predicate satisfies the following three properties for all sentences \( \varphi \) and \( \psi \):
(1) If $\text{PA} \vdash \varphi$, then $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi^\frown)$.  
(2) If $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi \rightarrow \psi^\frown)$, then $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi^\frown) \rightarrow \text{Prov}_{\text{PA}}(\psi^\frown)$.  
(3) $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi^\frown) \rightarrow \text{Prov}_{\text{PA}}(\varphi^\frown)$.  

(7a) Fix a sentence $\varphi$. Find a sentence $\theta$ so that $\text{PA} \vdash \theta \leftrightarrow (\text{Prov}_{\text{PA}}(\theta^\frown) \rightarrow \varphi)$.  

(7b) Assuming $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi^\frown) \rightarrow \varphi$, show that $\theta$ is provable from $\text{PA}$. Use this to derive Löb’s theorem that if $\text{PA} \vdash \text{Prov}_{\text{PA}}(\varphi^\frown) \rightarrow \varphi$, then $\text{PA} \vdash \varphi$.  

(7c) Give a sentence $\varphi$ so that $\text{PA}$ does not prove $\varphi \rightarrow \text{Prov}_{\text{PA}}(\varphi^\frown)$.  

Problem 8. A structure $\mathcal{A}$ is computable if its universe $A$ is a computable subset of $\omega$, and its functions, relations, and constants are uniformly computable. (Or equivalently, the atomic diagram of $A$ is computable.) Show there is no computable model of ZFC.