[1] (5 Pts.) Let $g(x)$ be a continuously differentiable function and consider the fixed point problem

$$x = g(x).$$

(a) What conditions on $g(x)$ and $\alpha, 0 < \alpha \leq 1$, will guarantee convergence of the iteration

$$x^* = g(x_n),$$

$$x_{n+1} = \alpha x^* + (1 - \alpha) x_n$$

to the solution $\bar{x}$ of the fixed point problem?

(b) Prove that under the conditions that you derived in (a) the solution $\bar{x}$ of the fixed point problem is unique.

[2] (5 Pts.) For a given value of $h > 0$ consider the two approximations to $f'(x)$

$$D_h f = \frac{f(x + h) - f(x)}{h}$$

$$D_{2h} f = \frac{f(x + 2h) - f(x)}{2h}$$

Derive the coefficients $\beta_1$ and $\beta_2$ so that the combination of approximations $\beta_1 D_h f + \beta_2 D_{2h} f$ is a second order approximation to $f'(x)$.

[3] (5 Pts.) Assume the points $\{x_i\}$, for $i = 1 \ldots n + 1$, are distinct. Prove that the polynomial of degree $\leq n$ that interpolates the data $\{(x_i, f(x_i))\}$ is unique.
[4] (10 Pts.) Consider the linear two-step numerical method for solving \( \frac{dy}{dt} = f(t, y) \),

\[
y_{i+2} = y_{i+1} + dt \left[ \frac{3}{2} f(t_{i+1}, y_{i+1}) - \frac{1}{2} f(t_i, y_i) \right].
\]

(a) Is this method consistent? Explain.
(b) What is the order of this method? Show your work.
(c) Does this method converge? Explain.
(d) Find a necessary and sufficient condition for the linear stability of the method (show your analysis, but without solving explicitly the obtained set of inequalities in the complex domain).

[5] (10 Pts.) Consider the hyperbolic equation

\[
u_t + u_x + 2u_y = 0
\]

for \( t > 0 \), \( (x, y) \) in the square \([-1, 1] \times [-1, 1]\), and initial data

\[
u(x, y, 0) = \varphi(x, y)
\]

(a) Boundary conditions on \( u \) are imposed to be zero on which sides of the square? Why?
(b) Set up a finite difference approximation which converges to the correct solution. Justify your answer.

[6] (10 Pts.) Consider the equation

\[
u_t = u_{xx}
\]

to be solved for \( t > 0 \), \( x \in [-1, 1] \), with periodic initial data

\[
u(x, 0) = \varphi(x), \quad \varphi(x + 2) \equiv \varphi(x)
\]

and \( u(x, t) \) periodic in \( x \) for \( t > 0 \). Give a fourth or higher order accurate convergent finite difference scheme. Justify your answer.
Consider the following problem in a domain $\Omega \subset \mathbb{R}^2$, with $\Gamma = \partial \Omega$:

$$
-\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u = f \text{ in } \Omega,
$$

$$
u = 0 \text{ on } \Gamma,
$$

where $\beta_1$ and $\beta_2$ are constants.

(a) Choose an appropriate space of test functions $V$, and give a weak formulation of the problem.

(b) For any $v \in V$, show that

$$
\int_{\Omega} \left( \beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v \right) dx = 0.
$$

(c) By analyzing the linear and bilinear forms, show that the weak formulation has a unique solution.

(d) Set up a convergent finite element approximation and discuss the linear system thus obtained.