DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.
Do all 7 problems.

Problems 1-3 are worth 5 points; problems 4-7 are worth 10 points.
All problems will be graded and counted towards the final score.
You have to demonstrate a sufficient amount of work on both groups of problems [1-3] and [4-7].

[1] (5 Pts.) The following code, implementing the bisection method, was used to find the roots of
\[ x^2 - 2x + 1 = 0 \]
using an initial interval of \( a = -1.0 \) and \( b = 0.0 \). The result was

Approximate root of \( x^2 - 2x + 1 \) is \(-0.0000009537\)

(a) Is this program running correctly, and if not, what is the cause of the problem?

```matlab
% bisect.m
fstring = 'x^2-2*x+1';  % target function
a = -1.0;                % left starting endpoint
b = 0.0;                 % right starting endpoint
eps = 1.0e-06;           % root error bound tolerance
c = (a+b)/2.0;           % midpoint = approximate root
nMax = 100;              % allow only nMax iterations
n = 0;
while(abs(b-a) > 2.0*eps) & (n < nMax)
    eval(['x = a;',fstring,'];']); fa = ans;  % evaluate the function at a
    eval(['x = c;',fstring,'];']); fc = ans;  % evaluate the function at c
    if(fa*fc <= 0)                 % a root lies in the left interval
        b = c;
    else                            % a root lies in the right interval
        a = c;
    end
    c = (a+b)/2.0;                 % midpoint = approximate root
    n = n+1;
end
sprintf(['Approximate root of ',fstring,' is %%.10f \
',c])
```
2 (5 Pts.) Gauss-Laguerre quadrature rules have the form

\[ \int_0^\infty x^\alpha e^{-x} f(x) \, dx \simeq \sum_{j=1}^N w_j f(x_j) \]

where \( \alpha \) is a constant.

(a) Consider using such formulas to create approximations to integrals of the form

\[ \int_0^\infty g(x) \, dx. \]

Give the nodes and weights, \( \tilde{x}_j \) and \( \tilde{w}_j \), (derived from the nodes and weights of the Gauss-Laguerre rule) that give rise to approximations of the form

\[ \int_0^\infty g(x) \, dx \simeq \sum_{j=1}^N \tilde{w}_j g(\tilde{x}_j) \]

(b) For what types of functions will the integration rules you derived in (a) be exact?

3 (5 Pts.) Consider the task of approximating a function \( f(x) \) by a linear combination of \( N \) functions \( q_k(x) \), \( k = 1 \ldots N \), e.g.

\[ f(x) \simeq \sum_{k=1}^N c_k q_k(x) \quad \text{for} \ x \in [0, 1] \]

(a) Give the equations that determine the \( c_k \)'s so that \( \| f(x) - \sum_{k=1}^N c_k q_k(x) \|_2 \) is minimized. (The norm is taken over the interval \([0, 1]\).)

(b) How would you solve the resulting equations?
[4] (10 Pts.) Consider the linear two-step method

\[ y_{n+2} - 3y_{n+1} + 2y_n = h \left[ \frac{13}{12} f(t_{n+2}, y_{n+2}) - \frac{5}{3} f(t_{n+1}, y_{n+1}) - \frac{5}{12} f(t_n, y_n) \right], \]

for solving \( y'(t) = f(t, y(t)), \ y(t_0) = y_0. \)

(a) Show that the order of the method is 2.

(b) Is this method convergent?

(c) How would the numerical scheme perform when applied to the simple example \( y'(t) = 0, \ y(0) = 1 \) with the initial conditions \( y_0 = 1 \) and \( y_1 \neq y_0 \) obtained using a one-step method in the presence of roundoff errors? Justify your answers.

[5] (10 Pts.) Consider the equation

\[ u_{tt} = au_{xx} + 2bu_{xy} + cu_{yy} \]

to be solved for \( t > 0, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 \) with initial data:

\[ u(x, y, 0) = u_0(x, y) \]
\[ u_t(x, y, 0) = u_1(x, y) \]

and periodic boundary conditions

\[ u(x + 1, y, t) = u(x, y, t) \]
\[ u(x, y + 1, t) = u(x, y, t) \]

(a) For what values of the constants \( a, b, c \) is this a well posed problem?

(b) Write a stable convergent finite difference scheme for this problem.

Justify your answers.
[6] (10 Pts.) Consider the nonlinear equation

\[ u_t + \left(\frac{u^2}{2}\right)_x = bu_{xx} \]

to be solved for \( t > 0, \ 0 \leq x \leq 1 \) with initial data

\[ u(x, 0) = u_0(x) \]

and periodic boundary conditions

\[ u(x + 1, t) \equiv u(x, t), \]

for \( b > 0 \), a positive constant

(a) Write a finite difference approximation to this problem that satisfies a maximum and minimum principle for all \( b > 0 \).

(b) As \( b \) goes to zero, what difficulties do you expect to see with solutions to the finite difference approximation

\[ \frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{(u_{i+1}^n)^2 - (u_{i-1}^n)^2}{4\Delta x} = \frac{b(u_{i+1}^n - 2u_i^n + u_{i-1}^n)}{(\Delta x)^2} \]

[7] (10 Pts.) Let \( \Omega \) be an open, bounded and connected subset of \( \mathbb{R}^2 \), with sufficiently smooth boundary. Consider the problem

\[ -\frac{\partial}{\partial x} \left( (1 + 2x^2 + 3y^4)u_x \right) - u_{yy} = f \text{ in } \Omega, \]

\[ (1 + 2x^2 + 3y^4)u_x n_x + u_y n_y + \lambda u = g \text{ on } \Gamma = \partial\Omega, \]

where \( f \in L^2(\Omega) \), \( g \in L^2(\Gamma) \), \( \vec{n} = (n_x, n_y) \) is the outward unit normal to \( \partial\Omega \), and \( \lambda \geq 0 \) is a constant.

(a) Give weak variational formulations of the problem, by considering the cases \( \lambda = 0 \) and \( \lambda > 0 \). Show that each of these formulations have one and only one solution (under additional conditions on \( u \), \( f \) or \( g \) if necessary, that you will specify).

(b) In the case \( \lambda > 0 \), describe a FE approximation using \( P_1 \) elements, and a set of basis functions such that the corresponding linear system is sparse. In particular show that the corresponding finite dimensional problem has a unique solution.

(c) What would be a standard error estimate for (b) with \( P_1 \) elements function of the meshsize \( h \) ? (assuming convexity and sufficient regularity of \( \Omega \) and of its boundary \( \Gamma \)).