DO NOT FORGET TO WRITE YOUR SID NO. ON YOUR EXAM.

There are 8 problems. Problems 1-4 are worth 5 points and problems 5-8 are worth 10 points. All problems will be graded and counted towards the final score.

You have to demonstrate a sufficient amount of work on both groups of problems [1-4] and [5-8] to obtain a passing score.

[1] (5 Pts.) Let \( f(0), f(h) \) and \( f(2h) \) be the values of a real valued function at \( x = 0, x = h \) and \( x = 2h \).

(a) Derive the coefficients \( c_0, c_1 \) and \( c_2 \) so that

\[ Df_h(x) = c_0 f(0) + c_1 f(h) + c_2 f(2h) \]

is as accurate an approximation to \( f'(0) \) as possible.

(b) Derive the leading term of a truncation error estimate for the formula you derived in (a).

[2] (5 Pts.) Let \( f(x) = \sqrt{\pi x} - \cos(\pi x) \).

(a) Show that the equation \( f(x) = 0 \) has at least one solution \( p \) in the interval \([0, 1]\).

(b) When using the Bisection method to approximate \( p \), how many iterations are necessary to solve \( \sqrt{\pi x} - \cos(\pi x) = 0 \) with accuracy \( 10^{-5} \) on \([0, 1]\)?

[3] (5 Pts.) If \( f(x) \) is sufficiently differentiable, then the error in approximations to \( I = \int_a^b f(x) \, dx \) obtained using the composite trapezoidal method, \( I_T(h) \), and the composite midpoint method, \( I_M(h) \), have asymptotic expansions of the form

\[ I - I_T(h) = -\frac{h^2}{12} \left( f'(b) - f'(a) \right) + O(h^4) \]

\[ I - I_M(h) = -\frac{h^2}{24} \left( f'(b) - f'(a) \right) + O(h^4) \]

where \( h \) is the mesh width.

(a) For a given value of \( h \), determine the combination of the values of \( I_T(h) \) and \( I_M(h) \) that results in an integral approximation with a higher order rate of convergence.

(b) What are the weights of the integration formula resulting from the combination you derived in (a)?
[4] (5 Pts.) Let \( f(x) \) be a real valued function and assume \( x_1 \neq 0 \). Prove that there exists a unique quartic polynomial \( q(x) \) so that

\[
q(0) = f(0), \quad q'(0) = f'(0), \quad q''(0) = f''(0), \quad q'''(0) = f'''(0),
\]
\[
q(x_1) = f(x_1).
\]

[5] (10 Pts.) Consider the linear system of ODE’s

\[
\frac{d\tilde{y}}{dt} = A \tilde{y}, \quad \tilde{y}(t_0) = \tilde{y}_0.
\]

where \( A \) is the \( N \times N \) symmetric tri-diagonal matrix

\[
\begin{pmatrix}
-2 & 1 & & & \\
1 & -2 & 1 & & \\
& 1 & -2 & 1 & \\
& & \ddots & \ddots & \ddots \\
& & & 1 & -2 & 1 \\
& & & & 1 & -2
\end{pmatrix}
\]

with \( h = \frac{1}{N+1} \).

(a) The eigenvalues of \( A \) are given by \( \lambda_k = -4 h^2 \sin^2\left(\frac{k\pi}{N+1}\right) \) \( k = 1 \ldots N \). Give a good estimate for the smallest value of the Lipschitz constant for the function \( \tilde{F}(\tilde{y}) = A\tilde{y} \) when using the \( \ell^2 \) norm on \( \mathbb{R}^N \).

(b) Assume approximate solutions of this system of equations are obtained for \( t \in [0,1] \) using Euler’s method with a uniform timestep of size \( \Delta t = \frac{1}{M} \). Give a derivation of an error bound for Euler’s method, and in particular, derive expressions for the constants, \( C_1 \) and \( C_2 \) appearing in an error bound of the form

\[
|\tilde{e}_n| \leq C_1 |\tilde{e}_0| + C_2 \Delta t \quad n = 1, 2, \ldots N
\]

where \( \tilde{e}_n = y^n - y(t_n) \). Show your work.

(c) Assuming \( \tilde{e}_0 = 0 \) and \( h = 0.1 \), give an estimate of \( \Delta t \) so that the magnitude of the error bound at \( t = 1.0 \) is less than \( 1.0 \times 10^{-3} \). (One can use the approximation that \( e \approx 10^{-4.35} \)).

(d) Is it necessary to use a timestep of the size determined in (c) to obtain an accurate solution? Explain your answer.
[6] (10 Pts.) Consider the initial value problem
\[ \frac{\partial u}{\partial t} + 1 + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 = c \frac{\partial^2 u}{\partial x^2} \]
to be solved for \(0 \leq x \leq 1\), \(t > 0\), with periodic boundary conditions in \(x\) and initial data
\[ u(x, 0) = f(x), \quad f(x) \text{ smooth}. \]
Here \(c\) is a positive constant.
(a) Construct a stable, convergent finite difference scheme for \(c > 0\) that remains convergent even as \(c \downarrow 0\). Hint: Differentiate the equation with respect to \(x\) and solve for \(\frac{\partial u}{\partial x} = v\), and use everything you know about the resulting equation for \(v\).
(b) Justify your answer.

[7] (10 Pts.) Consider the scalar second order equation for \(u(x, t)\)
\[ a u_{tt} + 2 b u_{xt} + c u_{xx} = 0 \]
to be solved for \(t > 0\), \(0 \leq x \leq 1\) with periodic boundary conditions in \(x\) and initial data
\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \]
with \(a, b\) and \(c\) constants and \(f(x)\) and \(g(x)\) smooth.
(a) For what values of \(a, b\) and \(c\) is this problem well posed?
(b) Devise a convergent finite difference scheme to create approximate solutions to this problem.
(c) Justify your answers.
Hint: For parts (b) and (c), one option might be to make this into an equivalent first order system of equations.
Consider the problem in two dimensions,

\[-\triangle u + u = f(x,y), \ (x,y) \in T,\]
\[u = 0, \ (x,y) \in T_1 \cup T_2,\]
\[\frac{\partial u}{\partial n} = h(x,y), \ (x,y) \in T_3,\]

where

\[T = \{(x,y) | x > 0, \ y > 0, \ x + y < 1\},\]
\[T_1 = \{(x,y) | y = 0, \ 0 < x < 1\},\]
\[T_2 = \{(x,y) | x = 0, \ 0 < y < 1\},\]
\[T_3 = \{(x,y) | x > 0, \ y > 0, \ x + y = 1\}.\]

(a) Find the weak variational formulation of the problem and verify the assumptions of the Lax-Milgram Lemma by analyzing the appropriate bilinear and linear forms (impose the weakest necessary assumptions on the functions \(f\) and \(h\)).

(b) Develop and describe the piecewise linear Galerkin finite element approximation of the problem and a set of basis functions such that the corresponding linear system is sparse. Show that this linear system has a unique solution. State the rate of convergence for the approximation.